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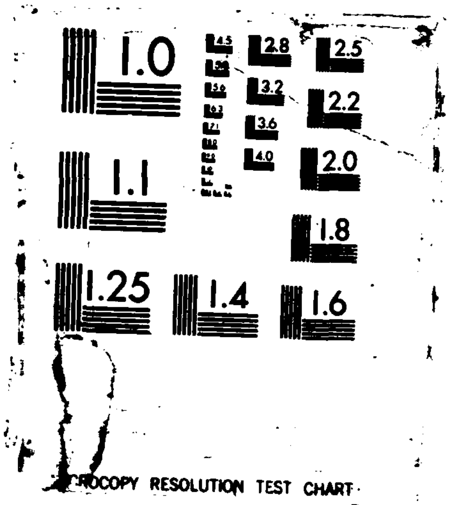
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A computer program written in FORTRAN 77 and recorded on diskettes compatible with an IBM PC computer. A Handbook was written containing: all expressions of stress, strains and displacements in closeform with an analysis of the manner how they were obtained and all the required theoretical references. A complete layout of program: a printed list of program: an instruction list for its utilization.

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STRESSES AND DISPLACEMENTS IN TWO, THREE AND FOUR LAYERED STRUCTURES SUBMITTED TO FLEXIBLE OR RIGID LOADS

CONTRACT DAJA45-86-M-0483

FINAL REPORT

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STRESSES AND DISPLACEMENTS IN TWO, THREE AND FOUR LAYERED SYSTEMS WITH FIXED BOTTOM

Introduction

This report deals with the mathematical aspects required for the establishment of a computer program able to calculate all stresses and displacements in two, three and four layered systems.

The materials of the different layers may be isotropic or cross-anisotropic.

The interface conditions cover all the cases from full friction to full slip included partial friction.

The bottom or the last layer is considered to be fixed (no vertical deflections).

The loads can either be flexible either rigid.

The report is based on :

- existing material : isotropic multilayer theory (BURMISTER, 1943) and anisotropic multilayer theory (VAN CAUWELAERT, 1983).
- original research work : interface conditions (fixed bottom, partial friction) and satisfactory convergency, thus complete accuracy, at the surface and in the first layer of the system, rigid load boundary condition.

This report contains three parts :

<u>Part 1</u> : a theoretical outline wherin the basic equations are given, the specific boundary conditions discussed and the particular numeral problems related to the accuracy at the surface and in the first layer, solved.

<u>Part 2</u> : the general mathematical analysis of the chosen numerical solution and a description of the programs and their utilization.

<u>Part 3</u> : (appendices) the detailled mathematical and algebrical analysises for the different considered cases : isotropic or anisotropic, full slip or not, flexible or rigid load.

The programs are written in FORTRAN 77 and run on IBM PC or all other compatible equipment.

PART 1. THE MULTILAYER SOLUTION

1.1 THE BASIC EQUATIONS AND FUNDAMENTAL HYPOTHESISES

The stresses and displacements in a multilayered system of homogeneous and isotropic layers subjected to a uniform vertical load applied over a circular area are obtained from the following stress function:

$$\phi = \int_o^\infty \frac{J_o(mr).F(m)}{m} [A_i e^{mz} - B_i e^{-mz} + zC_i e^{mz} - zD_i e^{-mz}]\ dm$$

Resulting stresses and displacements are given by:

$$\sigma_z = \int_o^\infty J_o(mr).F(m)[A_i m^2 e^{mz} + B_i m^2 e^{-mz}$$

$$- C_i m(1 - 2\mu_i - mz)e^{mz} + D_i m(1 - 2\mu_i + mz)e^{-mz}]\ dm$$

$$\sigma_r = - \int_o^\infty J_o(mr).F(m)\ [A_i m^2 e^{mz} + B_i m^2 c^{-mz}$$

$$+ C_i m(1 + 2\mu_i + mz)e^{mz} - D_i m(1 + 2\mu_i - mz)e^{-mz}]\ dm$$

$$+ \int_o^\infty \frac{J_1(mr).F(m)}{mr} [A_i m^2 e^{mz} + B_i m^2 e^{-mz}$$

$$+ C_i m(1 + mz)e^{mz} - D_i m(1 - mz)e^{-mz}]\ dm$$

$$\sigma_\theta = - \int_o^\infty J_o(mr).F(m)[C_i m.c^{mz} - D_i m.e^{-mz}].2\mu_i.dm$$

$$- \int_o^\infty \frac{J_1(mr).F(m)}{mr} [A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i m(1 + mz)e^{mz}$$

$$- D_i m(1 - mz)e^{-mz}]\ dm$$

$$\tau_{rz} = - \int_o^\infty J_1(mr).F(m)\ [A_i m^2 e^{mz} - B_i m^2 e^{-mz}$$

$$+ C_i m \, (2\mu_i + mz)e^{mz} + D_i m(2\mu_i - mz)e^{-mz}] \, dm$$

$$w = \frac{1+\mu_i}{E_i}\int_o^\infty \frac{J_o(mr).F(m)}{m} \, [A_i m^2 e^{mz} - B_i m^2 e^{-mz}$$

$$- C_i m \, (2 - 4\mu_i - mz)e^{mz} - D_i m \, (2 - 4\mu_i + mz)e^{-mz}] \, dm$$

$$u = -\frac{1+\mu_i}{E_i}\int_o^\infty \frac{J_1(mr).F(m)}{m} \, [A_i m^2 e^{mz} + B_i m^2 c^{-mz}$$

$$+ C_i m \, (1 + mz)e^{mz} - D_i m \, (1 - mz)e^{-mz}] \, dm$$

where

a = radius of the uniformly loaded circular area

p = intensity of the uniform vertical load

r = horizontal distance from the axis in a cylindrical coordinate system

z = depth

σ_z = vertical stress

σ_r = horizontal radial stress

σ_θ = circumferential stress

τ_{rz} = shear stress

w = vertical deflection

u = radial (horizontal) displacement

E_i = Young modulus of a given layer

μ_i = Poisson's ratio of a given layer

A_i, D_i = unknown parameters to be determined by the boundary conditions

J_o = Bessel function of the first kind of order zero

m = integration parameter

In the case of a cross-anisotropic body the stress function is:

$$\phi = \int_o^\infty J_o(mr).F(m)/m\ [A_i e^{mz} - B_i e^{-mz} + C_i e^{ms_i z} - D_i e^{-ms_i z}]\ dm$$

This stress function differs fundamentally from the isotropic case such that the two cases must be handled different.

The stresses and displacements are given by

$$\sigma_z = \int_o^\infty J_o(mr).F(m)\ [n_i(1 + \mu_i)\ (A_i m^2 e^{mz} + B_i m^2 c^{-mz})$$

$$+ n_i(n_i + \mu_i)\ (C_i s_i n^2 e^{-s_i mz} + D_i S_i m^2 e^{-s_i m2})]\ dm$$

$$\sigma_r = - \int_o^\infty J_o(mr).F(m)\ [n_i(1 + \mu_{i)}\ (A_i m^2 e^{mz} + B_i m^2 e^{-mz})$$

$$+ \frac{n_i(n_i - \mu_i^2)}{n_i - \mu_i}(C_i s_i m^2 e^{s_i mz} + D_i s_i m^2 e^{-s_i mz})]\ dm$$

$$+ \int_o^\infty \frac{J_1(mr)F(m)}{mr}\ n_i(1 + \mu_i)[A_i m^2 e^{mz} + B_i m^2 e^{-mz}$$

$$+ C_i s_i m^2 e^{s_i mz} + D_i s_i m^2 e^{-s_i mz}]\ dm$$

$$\sigma_\theta = - \int_o^\infty J_o(mr).F(m)\ \ [C_i s_i m^2 e^{s_i mz} + D_i s_i m^2 e^{-s_i mz}]\ \frac{n_i \mu_i (1-n_i)}{n_i - \mu_i}$$

$$- \int_o^\infty \frac{J_1(mr).F(m)}{mr}\ [A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i s_i m^2 e^{s_i mz}$$

$$+ D_i s_i m^2 e^{-s_i mz}]\ n_i(1+\mu_i)dm$$

$$\tau_{rz} = - \int_o^\infty J_1(mr).F(m)\ [n_i(1 + \mu_{i)}\ (A_i m^2 e^{mz} - B_i m^2 e^{-mz})$$

$$+ n_i s_i (n_i + \mu_i) \left(C_i s_i m^2 e^{s_i mz} - D_i s_i m^2 e^{-s_i mz}\right)\Big] dm$$

$$w = \frac{1 + \mu_i}{E_i} \int_o^\infty \frac{J_o(mr).F(m)}{m} \Big[n_i(1 + \mu_i) \quad (A_i m^2 e^{mz} - B_i m^2 e^{-mz})$$

$$+ \frac{n_i s_i (n_i + \mu_i^2)}{(1+\mu_i)} (C_i s_i m^2 e^{s_i mz} - D_i s_i m^2 e^{s_i mz})\Big] dm$$

$$u = \frac{(1 + \mu_i)\, n_i (n_i + \mu_i)}{E_i} \int_o^\infty \frac{J_1(mr).F(m)}{m} \Big[A_i m^2 e^{mz} + B_i m^2 e^{-mz}$$

$$+ C_i s_i m^2 e^{s_i mz} + D_i s_i m^2 e^{-s_i mz}\Big] dm$$

where

$n_i = Evi/Ehi$ is the degree of anisotropy expressed as the ratio between the vertical and the horizontal Youngs moduli

$s_i = \dfrac{\sqrt{n_i - \mu_i^2}}{n_i^2 - \mu_i^2}$ is the index of anisotropy.

The anisotropic relations are established by the assumption that G_{rz} the shear modulus in the vertical plane, is related to the other elastic constants by (BARDEN, 1963; VAN CAUWELERT, 1983):

$$\frac{1}{G_{rzi}} = \frac{1 + n_i + 2\mu_i}{E_i}$$

and that ν, Poisson's ratio in the horizontal plane, is related to Poisson's ratio in the vertical plane by (EFTIMIE, 1973; VAN CAUWELAERT, 1983):

$$\nu_i = \mu_i/n_i$$

1.2. THE BOUNDARY CONDITIONS

1.2.1 The Surface Conditions

The surface conditions are expressed by an adequate value of the Kernel $F(m)$ in the stress function.

In the case of a uniform distributed load over a flexible area the kernel is

$$F(m) = paJ_1(ma)$$

where,

J_1 = Bessel function of the first kind of order one.

Indeed the expression for the vertical stress is then at the surface.

$$\sigma_z = pa\int_o^\infty J_o(mr)\, J_1(ma)\, dm$$

$$= p \text{ for } r < a$$

$$= p/2 \text{ for } r = a$$

$$= o \text{ for } r > a$$

In the case of a load distributed over a rigid area the kernel is

$$F(m) = \frac{pa}{2} \sin\,(am)$$

at the surface

$$\sigma_z = \frac{pa}{2} \int_o^\infty J_o(mr) \sin(am)\, dm$$
$$= \frac{pa}{2} (a^2-r^2)^{-1/2} \quad \text{for } r < a$$
$$= 0 \quad \text{for } r > a$$

The total load

$$P = \int_o^{2\pi} \int_o^a \sigma_z r dr d\theta = \pi p a^2$$

is equal to the applied load.

Also at the surface, the deflection is

$$w = \frac{(1-\mu_1^2)}{E_1} pa \int_o^\infty \frac{J_o(m) \sin(am)}{m} dm$$
$$= -\frac{2(1-\mu_1^2)}{E_1} \frac{pa}{\pi} \quad \text{for} \quad r < a$$

The deflection is constant under the load.

1.2.2 <u>The Interface Conditions</u>.

Let us consider an n-layered system, consisting of (n-1) layers each having a finite thickness supported by a semi-infinite body. There is associated with each layer a stress function Φ_i (A_i B_i C_i D_i) with 4 unknown parameters such that the total of unknown parameters is 4n. Two of the parameters depend on the loading surface conditions.

$$\sigma_z = f(p) \quad \text{for } r < a$$

$$\tau_{rz} = 0$$

At infinite depth the stresses and displacements must vanish and thus A_n and C_n = o. As a result, there are 4 (n-1) parameters to be determined with 4 conditions at each interface. The four interface conditions are addressed by imposing the conditions that the layers maintain contact and that the vertical stresses (σ_z), shear stresses (τ_{rz}) and vertical displacements (w) at the bottom of each layer and at the top of the underlying layer are equal. The fourth interface condition, horizontal displacements (u), depends on the relative adhesion at the interface between the layers. The two extremes of adhension are

- full continuity, expressed by setting the horizontal displacement (u) on each side of the interface equal.
- frictionless interface expressed by considering the interface as a principal plan which results in the shear stresses equaling zero.

Partial adhesion can be expressed by:

$$u_i = Ku_{i+1}$$

with

$K \varepsilon$ [0,inf]
When K = 1, one has full continuity
K ≠ 1, one has partial continity

Zero friction then becomes a separate case, for which another program has to

be written.

1.2.3 The fixed bottom condition

The boundary conditions discussed in the previous paragraph indicate that the last layer of the multilayer is considered as a semi-infinite body. One can also consider the case of a multilayer system resting on a rigid body, such that vertical displacements vanish at the contact face with the rigid body. This problem shall be designated as the fixed bottom condition.

In addition, through the general solutions of the compatibility equation in multilayer layer theory, we can get w=o at the desired depth and determine the corresponding values of the parameters A_n, B_n, C_n, D_n. In the isotropic case, the parameter C_n must then be put equal to zero, while the parameter A_n is determined by condition w=o. In the anisotropic case, either the parameter A_n must be zero when s<1, or the parameter C_n must be zero when s>1 Using criteria that a minimum influence on the surface deflection is desireable and the fact that conditions u = o and T_{rz} = o have less physical sense, we retain the condition w = o as the most reasonable rigid bottom condition. An added benefit of this selection is that the condition w=o is the easiest to account for mathematically.

1.2.4 Influence of a fixed bottom condition on the numerical computation.

The deflection at the surface of a two layered system, without fixed bottom, was solved by BURMISTER (1945)

$$w = -pa \frac{2(1-\mu_1^2)}{E_1} \int_o^\infty \frac{J_o(mr).J_1(ma)}{m}$$

$$\left|\frac{1 + 4Kmhe^{-2mh} - KLe^{-4mh}}{1 - (L + K + 4km^2h^2)e^{-2mh} + KLe^{-4mh}}\right| dm$$

where

$$K = \frac{1 - n}{1 + n(3-4\mu_1)} \quad L = \frac{(3-4\mu_2) - n(3-4\mu_1)}{(3-4\mu_2) + n}$$

$$n = \frac{E_2\ (1 + \mu_1)}{E_1\ (1 + \mu_2)}$$

which, to avoid convergency problems during the numerical computation (see art 1.3.3), is written as:

$$w = -pa\frac{2(1 - \mu_1^2)}{E_1} \int_o^\infty \frac{J_o(mr).J_1(ma)}{m} dm$$

$$-pa\ \frac{2(1 - \mu^2)}{E_1} \int_o^\infty \frac{J_o(mr).J_1(ma)}{m}$$

$$\left|\frac{(L + K + 4Kmh + 4Km^2h^2)e^{-2mh} - 2KLe^{-4mh}}{1 - (L + K + 4Km^2h^2)e^{-2mh} + KLe^{-4mh}}\right| dm$$

$$= W_1 + W_2$$

The deflection with depth on the axis of the loaded area is computed ($r = 0$, $J_o\ (mr) = 1$) with $E_1 = 1{,}000$, $E_2 = 100$, $\mu_1 = \mu_2 = 0.5$, $a = 10$, $h = 10$, $p = 1$. The value of W_1 is obtained analytically (WATSON, 1966)

$$w_1 = -pa \frac{2(1 - \mu_1^2)}{E_1} \int_o^\infty \frac{J_1(ma)}{m} dm = -pa \frac{2(1 - \mu_1^2)}{E_1} = -0.015$$

The value of W_2 is obtained by numerical integration for different values of the integration parameter m:

m	w_2	w
0.5	- 0.0582	- 0.0732
0.1	- 0.0580	- 0.0730
0.05	- 0.0580	- 0.0730
0.02	- 0.0580	- 0.0730
0.01	- 0.0580	- 0.0730

From this data, the deflection is correct for values of m=0.1 or smaller. For the case of a rigid bottom at a depth H the value of the deflection at the surface is given, with $\mu = 0.5$, by

$$w = - \frac{1.5pa}{E} \int_o^\infty \frac{J_1(ma)}{m} \cdot \frac{1 - e^{-2mH}}{1 + (1 + 2mH)e^{-2mH}} dm$$

In this expression, for m = 0, the value of the integrand is equal to zero, while in the previous case the value was:

$$-pa \frac{2(1 - \mu_1^2)}{E_1} \cdot \frac{1}{2} \cdot \frac{L + K - 2KL}{1 - L - K - KL}$$

By splitting the integral the expression becomes:

$$w = -\frac{1.5pa}{E}\int_o^\infty \frac{J_1(ma)}{m}\,dm$$

$$+\frac{1.5pa}{E}\int_o^\infty \frac{J_1(ma)}{m}\cdot\frac{2(1+mH)e^{-2mH}}{1+(1+2mH).e^{-2mH}}\,dm$$

With $p = 1$, $a = 10$ and $E - 1{,}000$, the value of $w_1 = -\ 0.015$; the values of w_2 as a function of m, are computed

- with H = 3a

m	w_2	w
0.5	+ 0.0036	- 0.0114
0.1	+ 0.0033	- 0.0117
0.05	+ 0.0032	- 0.0118
0.02	+ 0.0032	- 0.0118

- with H - 1,000a

m	w_2	w
0.5	+ 0.0833	+ 0.0683
0.1	+ 0.0167	+ 0.0017
0.05	+ 0.0083	- 0.0067
0.02	+ 0.0033	- 0.0117
0.01	+ 0.0017	- 0.0133

For the case of a rigid bottom at H = 3a, the deflection is correct for values of m = 0.05 or smaller but for the case of a rigid bottom H = 1,000a, the

deflection is not correct for all the values m. In the latter case (H = 1000a), the problem is analogious to the semi-infinite layer case where w = -0.01.

For m = 0, the function f(m), in 1.5 pa/E $\int_0^\infty F(m)\,dm$, is equal zero: but at a value of m = 0+, the function is equal to its semi-infinite body value.

For comparison, the function is given below for various values of m for a semi-infinite body and for a deep rigid bottom.

m	f(m) semi-infinite	f(m) fixed bottom
0.00	0.5	0
0.01	0.499999	0.499999
0.02	0.499975	0.499975
0.05	0.499844	0.499844
0.10	0.499375	0.499375

In choosing a value too high for m, an important part of the deflection is neglected. The error is significant for the deep rigid bottom condition. As a result, the rigid bottom case must be handled carefully, while an average value of m = 0.1 gives correct results for the semi-infinite case.

1.3. PARTICULAR NUMERICAL PROBLEMS

Several numerical problems arise when accuracy is desired, whatever the location at which stresses and displacements are to be computed. In all cases the problems of accuracy can only be solved where the relations for stresses

and displacements are available, although partially, in closed form. Careful attention to specific terms within the general expressions can help to improve the accuracy of calculations.

1.3.1. The full slip interface condition

As was pointed out the zero friction or frictionless interface condition is a separate case that will be addressed. The value of any stress or displacement is obtained from one of the above mentioned relations. Let us consider, for example, the vertical stress in the i-th layer of an isotropic layer:

$$\sigma_z = pa \int_0^\infty J_0(mr).J_1(ma)\ [A_i m^2 e^{mz} + B_i m^2 e^{-mz} - C_i m(1-2\mu_i-mz)e^{mz} + D_i m(1-2\mu_i+mz)e^{-mz}]\ dm$$

Solution of this relation by numerical integration requires a value of m from o to a value high enough to ensure convergency. In the case of n layers and rigid bottom the parameters A_i,B_i,C_i and D_i must be determined for each value of m based on boundary conditions and a system of (4n - 1) equations with (4n - 1) unknowns.

Early solutions of this problem involved inverting the matrix of the (4n - 1) unknowns. However the inversion procedure leads, in some cases, to difficulties because of the presence of negative exponents tending to zero in the determinant of the denominator. Other programs have tried to avoid the inversion problem by using a trial. The procedure involves selecting values for B_n and D_n and solving the system of (4n - 1) equations. The solution is evaluated on how well the surface conditions are met. A second pair of values

for B_n and D_n is selected and a new solution obtained procedure. Since the process is linear, a good estimate of B_n and D_n can be made by linear interpolation. The difficulty lies in the appropriate choice of the values of B_n and D_n to ensure a numerically correct interpolation.

However, neither of the approaches are appropriate for all cases of a frictionless interface. The vertical deflectin at the surface of a two layer, the thickness of the first layer being H, is given by

$$w = pa\frac{1+\mu_1}{f1}\int_o^\infty \frac{J_o(mr).J_1(ma)}{m}\,[A_1 m^2 c^{-mH} - B_1 m^2 e^{mH}$$

$$- [2-4\mu_1+mH)\,C_1 me^{-mH} - (2-4\mu_1-mH)\,D_1 me^{mH}]\,dm$$

In the case of a frictionless interface, this relation becomes (BURMISTER, 1945)

$$w = -\,pa\frac{2(1-\mu_1^2)}{E_1}\int_o^\infty \frac{J_o(mr).J_1(ma)}{m}\,.$$

$$[\frac{Fe^{2mH} - (2F-1-2mH) - (1-F)e^{-2mH}}{Fe^{2mH} + (2F-1)2mH - (1+2m^2H^2) + (1-F)e^{-2mH}}]\,dm$$

where

$$F = \frac{(1-\mu_2) + n(1-\mu_1)}{2(1-\mu_2)} \quad \text{and} \quad n = \frac{E_2}{E_1}.\frac{(1+\mu_1)}{1+\mu_2}$$

For $m = 0$ the term in brackets becomes indefinite (0/0). This condition is not singificant when computing stresses, because the product of the Bessel functions is also zero at the origin ($J_o(mr).J_1(ma) = 0$ for $m = 0$.) However, in the case of deflection the

$$\lim_{m\to 0} \frac{J_1(ma)}{m} = \frac{a}{2}$$

As a result it is necessary to determine the term in brackets for $m = 0$ and thus to have it in a sufficiently closed form to be able to compute it.

1.3.2. Over and underflow problems

In the numerical integration procedure m varies from o to a value high enough to ensure convergency. Stated differently, the integration procedure can be stopped when the terms of the series become small enough so as not to have an influence on the final result. The value of m to achieve convergence may be as high as 20 to 30. The nature of convergency can be examined by considering the two-layer problem developed in the preceeding paragraph.

The parameters C_1 and D_1, from which the values of all the other parameters can be deduced, are given by:

$$C_1 = \frac{[(1-F+mH)e^{mH} - (1-F)e^{-mH}]}{Fe^{2mH} + (2F-1).2mH - (1+2m^2H^2) + (1-F)e^{-2mH}}$$

$$D_1 = \frac{[Fe^{mH} - (F-mH)e^{-mH}]}{Fe^{2mH} + (2F-1).2mH - (1+2m^2H^2) + (1-F)e^{-2mH}}$$

These expressions can be examined in terms of H/a, where a is the radius of the circular loaded area. At an H/a value of 5, a memory overflow will occur for values of m above 10 as a result of such exponential terms such as $e^{mH/a}$ and $e^{2mH/a}$. However, this problem can be overcome by dividing both numerator and denominator by e^{2mH}.

Expressing the parameters in this form will result in an underflow. However, most computers have a routine that sets variables subjected to underflow equal to zero. If such a routine does not exist, it can be build into the program.

By using the transformed relations for C_1 and D_1 convergency occurs quickly and in a completely safe way (i.e. the numerators both tend to zero, while the denominator tends to a constant F).

This can be obtained automatically for the two layer system by writing the boundary conditions at the surface (z = -H) as follows:

$$A_1 e^{-3mH} + B_1 e^{-mH} - C_1(1-2\mu_1 + mH)e^{-3mH} + D_1(1-2\mu_1-mH)e^{-mH} = e^{-2mH}$$

1.3.3. Stresses and Displacements at the Surface

Convergency is slow when surface stresses and displacements are computed because the parameters B_1 and D_1 content at this level a constant term in their numerators.

We use the surface conditions to express B_1 and D_1 in function of A_1 and C_1.

Each relation is then split into two parts: an integral of a product of Bessel functions which is computed analytically and an integral containing only negative exponents in the numerator which ensure normal convergency. In the case of a uniform distributed load the analytical integrals are:

$$\int_o^\infty J_o(mr)\, J_1(ma)\, dm = \frac{1}{a} \quad \text{for} \quad r < a$$
$$= \frac{1}{2a} \quad \text{for} \quad r = a$$
$$= o \quad \text{for} \quad r > a$$

$$\int_o^\infty J_1(mr)\, \frac{J_1(ma)}{mr}\, dm = \frac{1}{2a} \quad \text{for} \quad r < a$$
$$= \frac{a}{2r^2} \quad \text{for} \quad r > a$$

$$\int_o^\infty J_1(mr)\, \frac{J_1(ma)}{m}\, dm = \frac{r}{2a} \quad \text{for} \quad r < a$$

$$= \frac{a}{2r} \quad \text{for} \quad r < a$$

$\int_o^\infty J_o(mr) \frac{J_1(ma)}{m} dm$ which is a special case

For $r = o$, one has $\int_o^\infty \frac{J_1(ma)}{m} dm = 1$

For $r = a$, one has $\int_o^\infty \frac{J_0(mr).J_1(ma)}{m} dm = 2/\pi$

For $r < a$, one has $\int_o^\infty \frac{J_0(mr).J_1(ma)}{m} dm = F\,(1/2,\ -\ 1/2;\ 1;\ \frac{r^2}{a^2})$

where F is the hypergeometric function of GAUSS:

$$F\,(a,\ b;\ c;\ z) = \sum_o^\infty \frac{(a)_n\ (b)_n}{n!\ (c)_n}\, z^n$$

$(a)_n = a\ (a+1)\ (a+2)\ \ldots\ (a+n-1)$

$(a)_o = 1$

For $r > a$ $\int_o^\infty \frac{J_0(mr).J_1(ma)}{m} dm = \frac{a}{2r} F\,(1/2,\ 1/2;\ 2;\ \frac{a^2}{r^2})$

In the case of a rigid load the analytical integrals are:

$$\int_o^\infty J_o(mr)\ \sin\,(ma)\ dm = (a^2-r^2)^{-1/2} \quad \text{for} \quad r < a$$

$$= o \quad \text{for} \quad r > a$$

$$\int_o^\infty J_1(mr)\ \frac{\sin\,(ma)}{mr}\ dm = \frac{1}{r^2}\ [a-(a^2-r^2)]^{1/2} \quad \text{for} \quad r < a$$

$$= \frac{a}{r^2} \quad \text{for} \quad r > a$$

$$\int_o^\infty J_1(mr)\ \frac{\sin\,(ma)}{m}\ dm = \frac{1}{r}\ [a-(a^2-r^2)^{1/2}] \quad \text{for} \quad r < a$$

$$= \frac{a}{r} \quad \text{for} \quad r > a$$

$$\int_o^\infty J_o(mr)\ \frac{\sin\,(ma)}{m}\ dm = \pi/2 \quad \text{for} \quad r < a$$

$$= \text{arctg}\ [a\ (r^2-a^2)^{-1/2}]$$

1.3.4 Accuracy in the First Layer

As with computation of stresses at the surface the numerical computation of stresses in the first layer or near the surface converges slowly.

Again we express the parameters B_1 and D_1, associated with small exponentials, in function of the parameters A_1 and C_1 and split the obtained relations in an integral of a product of Bessel functions and an exponential (known as an LIPSCHITZ-HANKEL integral) which has to be treated in a particular way and an integral with normal numerical convergency.

Infinite LIPSCHITZ-HANKEL integrals can be transformed in finite and thus integrable, integrals when the Bessel indices are identical.

In the case of a uniform distributed load, a general solution can be achieved by considering the stresses and displacements in a semi-infinite body submitted to a force P.

With $P = p\pi a^2$

$$\lim_{a \to o} pa \int_o^\infty J_o(mr)\, J_1(ma)\, dm = \frac{P}{2\pi} \int_o^\infty J_o(mr)\, dm$$

Thus we may write for example

$$pa \int_o^\infty J_o(mr)\, J_1(ma)\, e^{-mz}\, dm = \frac{p}{2\pi} \int_o^{2\pi} \int_o^a \int_o^\infty \rho\, J_o(m\rho)\, e^{-mz}\, d\rho\, d\theta\, dm$$

where $\rho = (a^2+r^2-2ar \cos\theta)$

In this case

$$\int_o^\infty J_o(m\rho)\ e^{-mz}\ dm = (\rho^2+z^2)^{-1/2}$$

so that the infinite integral

$$pa \int_o^\infty J_o(mr)\ J_1(ma)\ e^{-mz}\ dm$$

is transformed in a finite integral

$$\frac{p}{2\pi} \int_o^{2\pi} \int_o^a \frac{\rho\ d\rho\ d\theta}{(\rho^2+z^2)^{1/2}}$$

One of the two integrals can be solved analytically. The required integrals are:

$$pa \int_o^\infty J_o(mr)\ J_1(ma)\ e^{-mz}\ dm = \frac{p}{2\pi} \int_{-a}^a \frac{2z(a^2-x^2)^{1/2}\ dx}{(z^2+x^2+r^2-2xr)\ (z^2+r^2+a^2-2xr)^{1/2}}$$

$$pa \int_o^\infty J_o(mr)\ J_1(ma)\ mze^{-mz}\ dm = \frac{p}{2\pi} \int_{-a}^a \frac{2z(a^2-x^2)^{1/2}\ dx}{(z^2+x^2+r^2-2xr)\ (z^2+r^2+a^2-2xr)^{1/2}}$$

$$+ \frac{p}{2\pi} \int_{-a}^a \frac{2z\ (a^2-x^2)^{1/2}\ dx}{(z^2+r^2+a^2-2xr)^{3/2}}$$

$$- \frac{p}{2\pi} \int_{-a}^a \frac{2z(r^2+x^2-2rx)\ (a^2-x^2)^{1/2}\ dx}{(z^2+x^2+r^2-2xr)\ (z^2+r^2+a^2-2xr)^{3/2}}$$

$$- \frac{p}{2\pi} \int_{-a}^a \frac{4z(r^2+x^2-2rx)\ (a^2-x^2)^{1/2}\ dx}{(z^2+x^2+r^2-2xr)^2\ (z^2+r^2+a^2-2xr)^{1/2}}$$

$$pa \int_0^\infty \frac{J_o(mr)\ J_1(ma)}{m} e^{-mz}\ dm = \frac{p}{2\pi} \int_{-a}^{a} \ln \frac{(r^2-2xr+a^2+z^2)^{1/2} + (a^2-x^2)^{1/2}}{(r^2-2xr+a^2+z^2)^{1/2} - (a^2-x^2)^{1/2}}\ dx$$

and following LIEBSCHITZ-HANKEL integrals

$$pa \int_0^\infty \frac{J_1(mr)\ J_1(ma)}{mr} e^{-mz}\ dm = p\frac{a^2}{\pi} \int_0^\pi \frac{1}{w^2} \left| 1 - \frac{z}{(z^2+w^2)^{1/2}} \right| \sin\theta\, d\theta$$

with $w^2 = a^2+r^2-2ar\cos\theta$

$$pa \int_0^\infty \frac{J_1(mr)\ J_1(ma)}{r} e^{-mz}\ dm = p\frac{a^2}{\pi} \int_0^\pi \frac{1}{(z^2+w^2)^{3/2}} \sin^2\theta\, d\theta$$

$$pa \int_0^\infty J_1(mr)\ J_1(ma)\ mze^{-mz}\ dm = 3p\frac{a^2rz^2}{\pi} \int_0^\pi \frac{1}{(z^2+w^2)^{5/2}} \sin^2\theta\, d\theta$$

In the case of a rigid load we expand either the Bessel function, $J_o(mr)$ or $J_1(mr)$, either the sine function, sin (ma), depending on the relative values of r and a, and solve the resulting series of integrals.

The required integrals are:

$$\int_0^\infty \frac{J_o(mr)\ \sin(ma)\ e^{-mz}}{m}\ dm =$$

$$\text{if} \quad r > a \quad \sum_0^\infty \frac{(-1)^n}{(2n+1)} \frac{a^{2n+1}}{(a^2+z^2)^{2n+1}}\ F \left| n+1/2, -n;\ 1;\ \frac{r^2}{r^2+z^2} \right|$$

$$\text{if} \quad r < a \quad \operatorname{arc\,tg}(a/z) + az \sum_{1}^{\infty} \frac{(-1)^n \; r^{2n} \; \Gamma\,(n+1/2)}{\Gamma(1/2) n! \; (a^2+z^2)^{1/2}} \; F \left| n+1, -n+1;\; 3/2;\; \frac{a^2}{a^2+z^2} \right|$$

$$\int_o^\infty J_o(mr) \sin(ma)\, e^{-mz}\, dm =$$

$$\text{if} \quad r > a \quad \sum_{o}^{\infty} \frac{(-1)^n \; a^{2n+1}}{(z^2+r^2)^{\frac{2n+3}{2}}} \; F \left| \frac{2n+3}{2},\; -n;\; 1;\; \frac{r^2}{r^2+z^2} \right|$$

$$\text{if} \quad r < a \quad 2a \sum_{o}^{\infty} \frac{(-1)^n \; r^{2n} \; \Gamma\,(n+3/2)}{\Gamma(1/2)\; n! \; (a^2+z^2)^{n+1}} \; F \left| n+1,\; -n;\; 3/2;\; \frac{a^2}{a^2+z^2} \right|$$

$$\int_o^\infty J_o(mr) \sin(ma)\, mz\, e^{-mz}\, dm =$$

$$\text{if} \quad r > a \quad az \frac{2z^2-r^2}{(z^2+r^2)^{5/2}} + 2z \sum_{1}^{\infty} \frac{(-1)^n \; a^{2n+1} (n+1)}{(z^2+r^2)^{\frac{2n+3}{2}}} \; F \left| \frac{2n+3}{2},\; -n-1;\; 1;\; \frac{r^2}{r^2+z^2} \right|$$

$$\text{if} \quad r < a \quad 4az^2 \sum_{o}^{\infty} \frac{(-1)^n \; r^{2n} \; (n+1)\; \Gamma\,(n+3/2)}{n! \; \Gamma(1/2)\; (z^2+a^2)^{n+2}} \; F \left| n+2,\; -n;\; 3/2;\; \frac{a^2}{a^2+z^2} \right|$$

$$\int_o^\infty \frac{J_1(mr) \sin(ma)\, e^{-mz}}{m}\, dm =$$

$$\text{if} \quad r > a \quad \frac{a}{r} - \frac{az}{r(r^2+z^2)^{1/2}} + \frac{rz}{2} \sum_{1}^{\infty} \frac{(-1)^n \; a^{2n+1}}{(r^2+z^2)^{\frac{2n+3}{2}}} \; F \left| \frac{2n+3}{2}, -n+1;\; 2;\; \frac{r^2}{r^2+z^2} \right|$$

$$\text{if} \quad r < a \quad a \sum_{o}^{\infty} \frac{(-1)^n \; r^{2n+1} \; \Gamma\,(n+3/2)}{(n+1)! \; \Gamma(1/2)\; (z^2+a^2)^{n+1}} \; F \left| n+1,\; -n;\; 3/2;\; \frac{a^2}{a^2+z^2} \right|$$

$$\int_o^\infty J_1(mr) \sin(ma)\, e^{-mz}\, dm =$$

$$\text{if} \quad r > a \quad \sum_{o}^{\infty} \frac{(-1)^n (n+1)\, a^{2n+1} \cdot r}{(r^2+z^2)^{\frac{2n+3}{2}}} F \left| \frac{2n+3}{2}, -n;\ 2;\ \frac{r^2}{r^2+z^2} \right|$$

$$\text{if} \quad r < a \quad 2 \sum_{o}^{\infty} \frac{(-1)^n r^{2n}\, az\, \Gamma\,(n+3/2)}{n!\ \Gamma(1/2)\ (a^2+z^2)^{n+2}} F \left| n+2, -n;\ 3/2;\ \frac{a^2}{a^2+z^2} \right|$$

$$\int_o^{\infty} J_1(mr)\ \sin(ma)\ mz\ e^{-mz}\ dm =$$

$$\text{if} \quad r > a \quad z \sum_{o}^{\infty} \frac{(-1)^n a^{2n+1} . r . z\ (2n+3)\ (n+1)}{(r^2+z^2)^{\frac{2n+5}{2}}} F \left| \frac{2n+5}{2}, -n;\ 2;\ \frac{r^2}{r^2+z^2} \right|$$

$$\text{if} \quad r < a \quad \frac{raz\ (3z^2-a^2)}{(a^2+z^2)^3} + 2raz \sum_{1}^{\infty} \frac{(-1)^n r^{2n}\ (2n+3)\ \Gamma(n+3/2)}{n!\ \Gamma(1/2)\ (a^2+z^2)^{n+2}} F \left| n+2, -n-1;\ 3/2;\ \frac{a^2}{a^2+z^2} \right|$$

PART 2: NUMERICAL RESOLUTION OF A FOUR LAYER SYSTEM

2.1 THE MATHEMATICAL ANALYSIS

As shown in art 1.3.1, the mathematical analysis must be different depending on the layer interface conditions.

2.1.1 Full or Partial Friction at the Interfaces

The case to be solved for first is full or partial friction at the interfaces. Boundary conditions for this case will be

- At the surface $\quad \sigma_z = p$

$\qquad \tau_{rz} = 0$

- At each interface $\quad \sigma_{zi} = \sigma_{zi+1}$

$\qquad \tau_{rzi} = \tau_{rzi+1}$

$$w_i = w_{i+1}$$

$$u_i = \lambda_i u_{i+1}$$

where λ_i is a factor for partial friction.

-At the bottom $w = 0$

A_4 or $C_4 = 0$ if rigid bottom

$A_4 = C_4 = 0$ if 4 - th layer is semi-infinite

Consider an isotropic four layer problem which will result in a system of 16 equations with 16 unknowns (A_1, B_1,....C_4, D_4). This system of equations is difficult to solve analytically in the same way that BURMISTER did for a two and three layered system. Using the same approach results in mathematical errors introduced in eliminating unknowns. As a result, it is necessary to modify the analysis procedure to solve accurately the numerical problems detailed in Part 1. The main objective of the approach is to obtain an expression for each unknown parameter consisting of a numerator containing negative exponents only and a denominator containing a constant term and negative exponents. During the integration procedure, when the variable tends to infinity, the numerator will then tend to zero and the denominator to a constant value. In the analysis the exponents must appear in close form. The factors multiplying the exponents may then be expressed in a more comprehensive form. Sequential steps in the mathematical analysis for a rigid bottom are as follows.

First Step

Replace in the boundary equations of the third interface the parameters A_4 and C_4 by their values obtained from the rigid bottom condition.

Second Step

Write all the interface conditions in matrix form.

- At the surface $MI\ (A_1\ B_1\ C_1\ D_1)^T = (1\quad 0)^T$

  where MI is a 2 x 4 matrix

- At the first interface $M_1(A_1\ B_1\ C_1\ D_1)^T = M_2(A_2\ B_2\ C_2\ D_2)^T$

where M_1 and M_2 are 4 x 4 matrices

- At the second interface $M_3(A_2\ B_1\ C_2\ D_2)^T = M_4(A_3\ B_3\ C_3\ D_3)^T$

where M_3 and M_4 are 4 x 4 matrices

- At the third interface $M_5(A_3\ B_3\ C_3\ D_3)^T = M_6(B_4\ D_4)^T$

where M_5 is a 4 x 4
and M_6 is a 4 x 2 matrix.

Third Step

Invert the matrices M_1, M_3 and M_5.

The system becomes

$$MI\ (A_1\ B_1\ C_1\ D_1)^T = (1\quad 0)^T$$
$$(A_1\ B_1\ C_1\ D_1)^T = M_1^{-1}\ .M_2(A_2\ B_2\ C_2\ D_2)^T$$
$$(A_2\ B_2\ C_2\ D_2)^T = M_3^{-1}\ .M_4(A_3\ B_3\ C_3\ D_3)^T$$
$$(A_3\ B_3\ C_3\ D_3)^T = M_5^{-1}.M_6(B_4\ D_4)^T$$

and finally

$$MI.M_1^{-1}M_2.M_3^{-1}.M_4.M_5^{-1}.M_6(B_4\ D_4)^T = (1\quad 0)^T$$

The product $MI.M_1^{-1}.M_2.M_3^{-1}.M_4.M_5^{-1}.M_6$ is a 2 x 2 matrix, so that we can write

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} B_4 \\ D_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and solve the resulting system:

$$B_4 = \frac{a_{22}}{a_{11} \cdot a_{22} - a_{21} \cdot a_{12}}$$

$$D_4 = \frac{-a_{21}}{a_{11} \cdot a_{22} - a_{21} \cdot a_{12}}$$

Utilizing the different matrix equations, we express then the other parameters as functions of B_4 and D_4. To obtain the exponents in close form, the matrices are split in such a way that the exponents can be factored out of the brackets. For example, in the isotropic case:

$$M_1^{-1} = - \frac{1}{4(1-\mu_1)} [e^{-x} \cdot M_{11} + e^{x} \cdot M_{12}]$$

$$M_2 = [e^{x} \cdot M_{21} + e^{-x} \cdot M_{22}]$$

$$M_3^{-1} = - \frac{1}{4(1-\mu_2)} [e^{-y} \cdot M_{31} + e^{y} \cdot M_{32}]$$

$$M_4 = [e^{y} \cdot M_{41} + e^{-y} \cdot M_{42}]$$

$$M_5^{-1} = - \frac{1}{4(1-\mu_3)} [e^{-z} \cdot M_{51} + e^{z} \cdot M_{52}]$$

$$M_6 = [e^{-u} \cdot M_{61} + e^{-z} \cdot M_{62}]$$

where $x = mh_1$

$y = m(h_1 + h_2)$

$z = m(h_1 + h_2 + h_3)$

$u = m(h_1 + h_2 + h_3 + 2h_4)$

h_i being the thickness of layer i.

The terms of all the matrices are in closed form and can be introduced as

input values for the numerical procedure. The terms of the resulting products are not expressed in closed form, however it is sufficient to know which columns or rows of each matrix contain only zeros. In the final product intermediate matrices disappear because they are identical zero. As a result, those matrices preceeded by positive exponents vanish. Subsequently, the values of all the unknown parameters except for the parameters B_1 and D_1 at the surface, can be expressed in terms of a numerator with only negative exponents and a denominator with a constant term and negative exponents.

At the surface, the values of the parameters B_1 and D_1 contain a constant term in both the numerator and the denominator followed by negative exponents which precludes obtaining satisfactory convergency. As an alternative, the parameters B_1 and D_1 are expressed, utilizing the surface conditions, as a function of the parameters A_1 and C_1. The resulting expression for a given stress or displacement becomes:

$$\sigma = pa \int_0^\infty J_0(mr).J_1(ma) \; [\; K + f_1(A_1) + f_2(C_1) \;] \; dm$$

where K is a constant term and $f_1(A_1)$ and $f_2(C_1)$ are functions of the parameters A_1 and C_1 which converge normally. Splitting the above relation into two parts results in:

$$\sigma_1 = pa \int_0^\infty J_0(mr).J_1(ma).\; K \; dm$$

which can be solved analytically

$$\sigma_2 = pa \int_0^\infty J_0(mr).J_1(ma) \; [\; f_1(A_1) + f_2(C_1) \;] \; dm$$

which can be solved numerically.

Near the surface the previous problem of B_1 and D_1 converging slowly arises too. Therefore, utilizing the surface conditions, the expressions for stresses and displacements are split into two parts. The first part, known as a LIPSCHITZ-HANKEL integral (paragraph 1.3.4), is solved analytically and the second part is solved numerically.

2.1.2 Full Slip at the First Two Interfaces and Full Friction at the Third Interfaces

A different procedure must be used to mathematically solve a layered system with frictionless conditions at some interfaces. The procedure described in previous paragraphs cannot be applied here. The boundary conditions for the case of full slip for the four layer system are:

- At the surface

$$\sigma_z = p$$

$$\tau_{rz} = o$$

- At the first two interfaces

$$\sigma_{zi} = \sigma_{zi+1} \qquad (1)$$

$$\tau_{rzi} = o \qquad (2)$$

$$\tau_{rzi+1} = o \qquad (3)$$

$$w_i = w_{i+1} \qquad (4)$$

- At the third interface

$$\sigma_{zi} = \sigma_{zi+1}$$

$$\tau_{rzi} = \tau_{rzi+1}$$

$$w_i = w_{i+1}$$

$$u_i = u_{i+1}$$

- At the bottom

$$w = o$$

$$A_4 \text{ or } C_4 = o \quad \text{for a rigid bottom}$$

The systems of equations at the first two interfaces cannot be expressed in matrix form because two of the four equations are homogeneous.

Sequential steps in the mathematical analysis are as follows:

First Step

Replace in the boundary equations of the third interface, the parameters A_4 and C_4 by their values obtained from the rigid bottom condition.

Second Step

Write the boundary equations at the third interface (full friction) in matrix form

$$(A_3\ B_3\ C_3\ D_3)^T = M_5^{-1} . M_6 (B_4\ D_4)^T$$

Third Step

Using the surface conditions, express A_1 and B_1 as a function of C_1 and D_1

$$(A_1\ B_1)^T = M_o (C_1\ D_1)^T$$

Using conditions (1) and (2) at the first interface replace A_1 and B_1 by their values and solve the system by expressing C_1 and D_1 as a function of A_2, B_2, C_2, D_2.

$$(C_1\ D_1)^T = M_1 (A_2\ B_2\ C_2\ D_2)^T$$

Using condition (4) at the first interface, replace A_1, B_1, C_1 and D_1 by their values expressed as functions of A_2, B_2, C_2, and D_2. Using conditions (3) and (4) at the first interface, express A_2 and B_2 as a function of C_2 and D_2

$(A_2\ B_2)^T = M_2(C_2\ D_2)^T$

Using condition (1) and (2) at the second interface, replace A_2 and B_2 by their values and solve the system by expressing C_2 and D_2 as functions of A_3, B_3, C_3, D_3.

$(C_2\ D_2)^T = M_3(A_3\ B_3\ C_3\ D_3)^T$

Using condition (4) at the second interface, replace A_2, B_2, C_2, and D_2 by their values expressed as functions of A_3, B_3, C_3 and D_3. Conditions (3) and (4) at the second interface are reduced to the following system

$M_4(A_3\ B_3\ C_3\ D_3)^T = (K\ \ o)^T$

where M_4 is a 2 x 4 matrix and K a function of the integration variable. Finally,

$M_4 \cdot M_5^{-1} \cdot M_6(B_4\ D_4)^T = (K\ \ o)^T$

a system which can be solved as in previous paragraph.

The same procedure is also utilized to express all the unknown parameters as functions of B_4 and D_4.

At this point a supplementary difficulty for the full slip case, regarding the expression of the vertical deflection must be considered. The vertical deflection is given by:

$$w = pa \frac{1+\mu_1}{E_1} \int_o^\infty \frac{J_o(mr) \cdot J_1(ma)}{m} f_1(A_1 B_1 C_1 D_1)\ dm$$

For $m = o$,

$$\lim_{m=o} \frac{J_o(mr).J_1(ma)}{m} = \frac{a}{2}$$

and

$$\lim_{m=o} f_i(A_iB_iC_iD_i) = \frac{0}{0}$$

which result in the above expression for the vertical deflection being undefined. However, at the rigid bottom w=0 for all values of the integrating parameter m_i, including m=o. Consequently,

At the bottom $f_4(A_4\ B_4\ C_4\ D_4) = o$

At the third interface $f_3(A_3\ B_3\ C_3\ D_3) = f_4(A_4\ B_4\ C_4\ D_4)$

At the second interface $f_2(A_2\ B_2\ C_2\ D_2) = f_3(A_3\ B_3\ C_3\ D_3)$

and, at the first interface $f_1(A_1\ B_2\ C_1\ D_1) = f_2(A_2\ B_2\ C_2\ D_2)$

Thus, in general for all layers $f_i(A_i\ B_i\ C_i\ D_i) = o$ for m = o.

Of course this is not any more true in the case of the last layer being a semi-infinite body. At the origin of integration (m=0) we then write the boundary conditions as follows:

$$\sigma_{z_1} = \sigma_{z_2} = \sigma_{z_3} = \sigma_{z_4} = \sigma_{z_5} = 1 \qquad B_5 + D_5\ (1 - 2\mu_5) = 1$$

$$\tau_{rz_1} = \tau_{rz_2} = \tau_{rz_3} = \tau_{rz_4} = \tau_{rz_5} = 0 \qquad -B_5 + 2\mu_5\ D_5 = 0$$

We obtain $B_5 = 2\mu_5$ and $D_5 = 1$

$$w_1 = w_2 = w_3 = w_4 = w_5$$

so that we find the origin term of each deflection by

$$\frac{1+\mu_1}{E_1}\left[(A_1 - B_1) - 2(1 - 2\mu_1)(C_1 - D_1)\right] = \frac{1+\mu_5}{E_5}\left[-B_5 + 2(1-2\mu_5) D_5\right]$$

$$= -\frac{1+\mu_5}{E_5} .2(1 - \mu_5)$$

2.2 THE NUMERICAL PROCEDURE

In general, numerical integration can be accomplished using Simpson's method of the form:

$$I (a,b) = \frac{h}{3}\left[f_o + 4f_1 + 2f_2 + \ldots + 4f_{2n-1} + f_{2n}\right]$$

where $[a,b]$ is the interval of integration of $I (a,b)$ subdivided in 2n equal segments of length h.

However, stresses and displacements are obtained by solution of expressions such as:

$$\sigma = pa \int_o^\infty J_o(mr).J_1(ma)\ f(m)\ dm$$

where the interval of integration goes until infinity. The numerical computation is interrupted when the value of the function $f (m,z)$ (in fact the values of all the parameter $A_i B_i C_i D_i$) becomes smaller than the imposed convergency level (i.e. $f_i(m,z) < \varepsilon$). The rate of change of the function, $f_i(mz)$, is significant for small values of the integrating parameters m and less significant for high values. A reasonable approach appears to be to increase the increment of m (i.e., h) for higher values for m. Using this technique, the above relation can then be written:

$$I\,[o,\infty] = \frac{h_1}{3}\,[f_o + 4f_1 + 2f_3 + f_4] \quad \text{for} \quad m < L_1$$
$$+ \frac{h_2}{3}\,[f_4 + 4f_5 + 2f_6 + 4f_7 + f_8] \quad \text{for} \quad L_1 < m < L_2$$
$$+ \frac{h_3}{3}\,[f_8 + 4f_9 + 2f_{10} + 4f_{11} + f_{12}] \quad \text{for} \quad L_2 < m$$

For practical reasons we take

$h_2 = 2h_1, \qquad h_3 = 2h_2, \ldots$

In all solutions, except for vertical deflection, $f_o = 0$ and the integral becomes:

$$I\,[o,\infty] = \frac{h}{3}\,[4f_1 + 2f_2 + 4f_3 + 2f_4] + \frac{h}{3}\,f_4$$
$$+ \frac{2h}{3}\,[4f_5 + 2f_6 + 4f_7 + 2f_8] + \frac{2h}{3}\,f_8$$
$$+ \frac{4h}{3}\,[4f_9 + 2f_{10} + 4f_{11} + 2f_{12}]$$

The main computation routine is

$$I\,[L_i,\ L_{i+1}] = \frac{2^i h}{3}\,[4f_{i+1} + 2f_{i+2} + \ldots\ 4f_{i+2n-1} + 2f_{i+2n}]$$

The length of the integration segment can be modified further with a value of

$\frac{2^i h}{3} \cdot f_{i+2n}$

The initial length h of the integration segment is chosen by the user and is multiplied by a factor of 2 when the values of f(mz) become smaller than 10^{-2}, 10^{-3}, 10^{-4} and 10^{-5}. The final convergency level is 10^{-6}.

For the solution of vertical deflection, $f_o \neq 0$ and is computed in an appropriate subroutine.

2.3 COMPUTER PROGRAM

A computer program has been prepared that runs efficiently on a personal computer. The main steps of the program can be resumed as follows.

2.3.1 Program Input Procedures

- Input of the data (by display or file) for loads and pavement structure

- Input of the coordinates (by display or file) of the points where stresses and displacements are to be computed.

- Choice of the length of the integration segment.

2.3.2 Computation Features

- Vertical deflection at the surface and in the first layer, for $m = o$.

- Parameters A_i B_i C_i D_i for each value of m.

- Bessel functions for each value of m.

- Stresses and displacements in cylindrical coordinates for each value of m.

- Stresses and displacements in cartesian coordinates for each value of m.

- Vectors containing the results using Simpson's rule.

- Convergency test for each f_{2k} function.

- When the final convergency of the numerical part is reached, computation of the analytical part of the solution for the surface and the first layer must be added (LIPSCHITZ-HANKEL Integrals).

2.3.3 Output Procedures

- stresses σ_x σ_y σ_z τ_{xy} τ_{xz} τ_{yz}

  displacements u_x v_y w_z

- principal stresses σ_1 σ_2 σ_3

  principal strains ε_1 ε_2 ε_3

  linear strains ε_x ε_y ε_z

2.3.4 Capabilities of the Program

The number of circular loads, with different radii and contact pressures, is limited to 20 and stresses and displacements can be computed at 30 locations in the horizontal plane and at each place at 30 depths (included 8 values at the surface and interfaces).

2.3.5 Software and Hardware Requirements

The program is written in FORTRAN 77 and runs on IBM PC's equipped with a 8087 Math Comprocessor and compatible equipment. The executable version requires 200 kilobytes of memory.

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APPENDIX 1

ALGEBRAICAL ANALYSIS OF ISOTROPIC LAYERED SYSTEMS WITH FIXED BOTTOM AND PARTIAL FRICTION CONDITIONS AT THE INTERFACES

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SYMBOLS

We write

$$A_i = A_i m^2 \qquad B_i = B_i m^2 \qquad C_i = C_i m \qquad D_i = D_i m$$

$$F_w = \frac{E_1(1+\mu_2)}{E_2(1+\mu_1)} \qquad F_u = \lambda_1 \frac{E_1(1+\mu_2)}{E_2(1+\mu_1)}$$

$$K_w = \frac{E_2(1+\mu_3)}{E_3(1+\mu_2)} \qquad k_u = \lambda_2 \frac{E_2(1+\mu_3)}{E_3(1+\mu_2)}$$

$$L_w = \frac{E_3(1+\mu_4)}{E_4(1+\mu_3)} \qquad L_u = \lambda_3 \frac{E_3(1+\mu_4)}{E_4(1+\mu_3)}$$

$$x = m H_1$$

$$y = m(H_1 + H_2)$$

$$z = m(H_1 + H_2 + H_3)$$

$$t = m(H_1 + H_2 + H_3 + H_4)$$

where H_1, H_2, H_3, H_4 are the thicknesses of the successive layers. The index 1 applies to the surface layer.

We shall successively analyse a two layered, a three layered and a four layered structure.

Chapter 1. The two layered structure.

1. The boundary conditions.

Boundary conditions at the surface ($z = 0$):

$$\sigma_z: \quad A_1 + B_1 - C_1(1-2\mu_1) + D_1(1-2\mu_1) = 1$$

$$\tau_{rz}: \quad A_1 - B_1 + C_1 \cdot 2\mu_1 + D_1 2\mu_1 = 0$$

Boundary conditions at the interface ($z = H_1$):

$$\sigma_z: \quad A_1 e^{x} + B_1 e^{-x} - C_1(1-2\mu_1-x)e^{x} + D_1(1-2\mu_1+x)e^{-x} =$$
$$A_2 e^{x} + B_2 e^{-x} - C_2(1-2\mu_2-x)e^{x} + D_2(1-2\mu_2+x)e^{-x}$$

$$\tau_{rz}: \quad A_1 e^{x} - B_1 e^{-x} + C_1(2\mu_1+x)e^{x} + D_1(2\mu_1-x)e^{-x} =$$
$$A_2 e^{x} - B_2 e^{-x} + C_2(2\mu_2+x)e^{x} + D_2(2\mu_2-x)e^{-x}$$

$$w: \quad A_1 e^{x} - B_1 e^{-x} - C_1(2-4\mu_1-x)e^{x} - D_1(2-4\mu_1+x)e^{-x} =$$
$$F_w\left[A_2 e^{x} - B_2 e^{-x} - C_2(2-4\mu_2-x)e^{x} - D_2(2-4\mu_2+x)e^{-x}\right]$$

$$u: \quad A_1 e^{x} + B_1 e^{-x} + C_1(1+x)e^{x} - D_1(1-x)e^{-x} =$$
$$F_u\left[A_2 e^{x} + B_2 e^{-x} + C_2(1+x)e^{x} - D_2(1-x)e^{-x}\right]$$

Boundary conditions at the bottom ($z = H_1 + H_2$):

$$w: \quad A_2 e^{y} - B_2 e^{-y} - C_2(2-4\mu_2-y)e^{y} - D_2(2-4\mu_2+y)e^{-y} = 0$$

$$C_2 = 0$$

2. Resolution of the system of 6 boundary equations.

We write the conditions at the interface in matrixform

$$M_1 (A_1 B_1 C_1 D_1)^T = M_2 (B_2 D_2)^T$$

and invert matrix M1

$$(A_1 B_1 C_1 D_1)^T = M_1^{-1} \cdot M_2 (B_2 D_2)^T$$

after having replaced A_2 by its value at the bottom

$$A_2 e^{x} = B_2 e^{-(2y-x)} + D_2(2-4\mu_2+y)\, e^{-(2y-x)}$$

$$M_1^{-1} = -\frac{1}{4(1-\mu_1)} e^{-x} \begin{pmatrix} -(1+x) & -(2-4\mu_1-x) & -(2\mu_1+x) & -(1-2\mu_1-x) \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-\frac{1}{4(1-\mu_1)} e^{x} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -(1-x) & (2-4\mu_1+x) & (2\mu_1-x) & -(1-2\mu_1+x) \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

$$M_2 = e^{-(2y-x)} \begin{pmatrix} 1 & (2-4\mu_2+y) \\ 1 & (2-4\mu_2+y) \\ F_w & F_w(2-4\mu_2+y) \\ F_u & F_u(2-4\mu_2+y) \end{pmatrix} + e^{-x} \begin{pmatrix} 1 & (1-2\mu_2+x) \\ -1 & (2\mu_2-x) \\ -F_w & -F_w(2-4\mu_2+x) \\ F_u & -F_u(1-x) \end{pmatrix}$$

$$(A_1, B_1, C_1, D_1)^T = -\frac{1}{4(1-\mu_1)} \left[e^{-x} M_{11} + e^{x} M_{12} \right] \left[e^{-(2y-x)} M_{21} + e^{-x} M_{22} \right] (B_2 \; D_2)^T$$

$$= -\frac{1}{4(1-\mu_1)} \left[e^{-2y} M_{1121} + e^{-2x} M_{1122} + e^{-2(y-x)} M_{1221} + M_{1222} \right] (B_2 \; D_2)^T$$

$$M_{1121} = \begin{vmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{vmatrix} \qquad M_{1122} = \begin{vmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{vmatrix} \qquad M_{1221} = \begin{vmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{vmatrix}$$

$$M_{1222} = \begin{pmatrix} 0 & 0 \\ F_1 & F_4 \\ 0 & 0 \\ -F_3 & F_2 \end{pmatrix} + x \begin{pmatrix} 0 & 0 \\ F_3 & (F_1 - F_2) \\ 0 & 0 \\ 0 & -F_3 \end{pmatrix} + x^2 \begin{pmatrix} 0 & 0 \\ 0 & F_3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ F_{21} & F_{22} \\ 0 & 0 \\ F_{41} & F_{42} \end{pmatrix}$$

$$F_1 = -3 + 4\mu_1 - F_u - 2\mu_1 (F_w - F_u)$$

$$F_2 = -1 - 2F_w + 4F_w \cdot \mu_2 - F_u$$

$$F_3 = F_w - F_u$$

$$F_4 = -1 + 6\mu_2 - 8\mu_1\mu_2 - 4\mu_1 F_w + 3\mu_1\mu_2 F_w + F_u - 2\mu_1 F_u$$

$$A_1 = -\frac{1}{4(1-\mu_1)} \Big\{ \big[e^{-2y} M_{1121}(1,1) + e^{-2x} M_{1122}(1,1) \big] B_2 + \big[e^{-2y} M_{1121}(1,2) + e^{-2x} M_{1122}(1,2) \big] D_2 \Big\}$$

$$= -\frac{1}{4(1-\mu_1)} \Big\{ \big[e^{-2(y-x)} M_{1121}(1,1) + M_{1122}(1,1) \big] B_2 e^{-2x} + \big[e^{-2(y-x)} M_{1121}(1,2) + M_{1122}(1,2) \big] D_2 e^{-2x} \Big\}$$

$$= -\frac{1}{4(1-\mu_1)} \big[A_{11} B_2 e^{-2x} + A_{12} D_2 e^{-2x} \big]$$

$$B_1 = -\frac{1}{4(1-\mu_1)} \Big\{ \left[M_{1221}(2,1)\, e^{-2(y-x)} + F_{21} \right] B_2 + \left[M_{1221}(2,2)\, e^{-2(y-x)} + F_{22} \right] D_2 \Big\}$$

$$= -\frac{1}{4(1-\mu_1)} \Big\{ \left[R_{21} + F_{21} \right] B_2 + \left[R_{22} + F_{22} \right] D_2 \Big\}$$

$$C_1 = -\frac{1}{4(1-\mu_1)} \Big\{ \left[M_{1121}(3,1)\, e^{-2(y-x)} + M_{1122}(3,1) \right] B_2 e^{-2x} + \left[M_{1121}(3,2)\, e^{-2(y-x)} + M_{1122}(3,2) \right] D_2 e^{-2x} \Big\}$$

$$= -\frac{1}{4(1-\mu_1)} \left[R_{31} B_2 e^{-2x} + R_{32} D_2 e^{-2x} \right]$$

$$D_1 = -\frac{1}{4(1-\mu_1)} \Big\{ \left[M_{1221}(4,1)\, e^{-2(y-x)} + F_{41} \right] B_2 + \left[M_{1221}(4,2)\, e^{-2(y-x)} + F_{42} \right] D_2 \Big\}$$

$$= -\frac{1}{4(1-\mu_1)} \Big\{ \left[R_{41} + F_{41} \right] B_2 + \left[R_{42} + F_{42} \right] D_2 \Big\}$$

$$(A_1\; B_1\; C_1\; D_1)^T = -\frac{1}{4(1-\mu_1)} \left[MR + MF \right] (B_2\;\; D_2)^T$$

We write the boundary conditions at the surface in matrix form

$$M_0\, (A_1\; B_1\; C_1\; D_1)^T = (1 \quad 0)^T$$

$$M_0 = \begin{vmatrix} 1 & 1 & -(1-2\mu_1) & (1-2\mu_1) \\ 1 & -1 & 2\mu_1 & 2\mu_1 \end{vmatrix}$$

$$M_0 \cdot \left[MR + MF \right] (B_2 \; D_2)^T = \left(-4(1-\mu_1) \quad 0 \right)^T$$

$$M_0 . MR = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \qquad M_0 MF = \begin{vmatrix} FF_{11} & FF_{12} \\ FF_{21} & FF_{22} \end{vmatrix}$$

$$A_{11} = R_{11} e^{-2k} + R_{21} - (1-2\mu_1) R_{31} e^{-2k} + (1-2\mu_1) R_{41}$$

$$A_{12} = R_{12} e^{-2k} + R_{22} - (1-2\mu_1) R_{32} e^{-2k} + (1-2\mu_1) R_{42}$$

$$A_{21} = R_{11} e^{-2k} - R_{21} + 2\mu_1 R_{31} e^{-2k} + 2\mu_1 R_{41}$$

$$A_{22} = R_{12} e^{-2k} - R_{22} + 2\mu_1 R_{32} e^{-2k} + 2\mu_1 R_{42}$$

$$FF_{11} = F_{21} + (1-2\mu_1) F_{41}$$

$$FF_{12} = F_{22} + (1-2\mu_1) F_{42}$$

$$FF_{21} = -F_{21} + 2\mu_1 F_{41}$$

$$FF_{22} = -F_{22} + 2\mu_1 F_{42}$$

We develop the matrix equation

$$(A_{11} + FF_{11}) B_2 + (A_{12} + FF_{12}) D_2 = -4(1-\mu_1)$$

$$(A_{21} + FF_{21}) B_2 + (A_{22} + FF_{22}) D_2 = 0$$

and solve the system

$$B_2 = -4(1-\mu_1) \frac{A_{22} + FF_{22}}{\nabla}$$

$$D_2 = 4(1-\mu_1) \frac{A_{21} + FF_{21}}{\nabla}$$

$$\nabla = A_{11}(A_{22} + FF_{22}) - A_{12}(A_{21} + FF_{21}) + FF_{11} A_{22} - FF_{12} A_{21} + FF_{11} . FF_{22} - FF_{12} . FF_{21}$$

We develop FF11.FF22 - FF12.FF21

$$FF_{11}.FF_{22} - FF_{12}.FF_{21} = F_1F_2 + F_3F_4$$

Al the other members of the denominator contain negative exponents.

The limit value of the denominator, for m = ∞ , is equal to a constant F1.F2 + F3.F4

3. Relations for the parameters.

The values of the parameters A_1 and C_1 are deduced from the matrix equation

$$(A_1\ B_1\ C_1\ D_1)^T = -\frac{1}{4(1-\mu_1)}\left[MR + MF\right](B_2\ D_2)^T$$

$$(A_1\ 0\ C_1\ 0)^T = -\frac{1}{4(1-\mu_1)}\ MR\ (B_2\ D_2)^T$$

$$A_1 = \frac{R_{11}(A22 + FF22)e^{-2x} - R_{12}(A21 + FF21)e^{-2x}}{\nabla}$$

$$A_1e^x = \frac{R_{11}(A22 + FF22)e^{-x} - R_{12}(A21 + FF21)e^{-x}}{\nabla}$$

$$C_1 = \frac{R_{31}(A22 + FF22)e^{-2x} - R_{32}(A21 + FF21)e^{-2x}}{\nabla}$$

$$C_1e^x = \frac{R_{31}(A22 + FF22)e^{-x} - R_{32}(A21 + FF21)e^{-x}}{\nabla}$$

The values of the parameters B_1 and D_1 are deduced from the surface conditions

$$B_1 = 2\mu_1 + (1-4\mu_1)A_1 + 4\mu_1C_1(1-2\mu_1)$$

$$D_1 = 1 - 2A_1 + C_1(1-4\mu_1)$$

$$B_2e^{-x} = -4(1-\mu_1)\ \frac{A22 + FF22}{\nabla}e^{-x}$$

$$D_2e^{-x} = -4(1-\mu_1)\ \frac{A21 + FF21}{\nabla}e^{-x}$$

Chapter 2. The three layered structure.

1. Supplementary boundary conditions.

Boundary conditions at the second interface ($z = H_1 + H_2$):

$$\sigma_z: \quad A_2 e^{y} + B_2 e^{-y} - C_2(1-2\mu_2-y)e^{y} + D_2(1-2\mu_2+y)e^{-y} =$$
$$A_3 e^{y} + B_3 e^{-y} - C_3(1-2\mu_3-y)e^{y} + D_3(1-2\mu_3+y)e^{-y}$$

$$\tau_{rz}: \quad A_2 e^{y} - B_2 e^{-y} + C_2(2\mu_2+y)e^{y} + D_2(2\mu_2-y)e^{-y} =$$
$$A_3 e^{y} - B_3 e^{-y} + C_3(2\mu_3+y)e^{y} + D_3(2\mu_3-y)e^{-y}$$

$$w: \quad A_2 e^{y} - B_2 e^{-y} - C_2(2-4\mu_2-y)e^{y} - D_2(2-4\mu_2+y)e^{-y} =$$
$$k_w\left[A_3 e^{y} - B_3 e^{-y} - C_3(2-4\mu_3-y)e^{y} - D_3(2-4\mu_3+y)e^{-y}\right]$$

$$u: \quad A_2 e^{y} + B_2 e^{-y} + C_2(1+y)e^{y} - D_2(1-y)e^{-y} =$$
$$k_u\left[A_3 e^{y} + B_3 e^{-y} + C_3(1+y)e^{y} - D_3(1-y)e^{-y}\right]$$

Boundary conditions at the bottom ($z = H_1 + H_2 + H_3$):

$$w: \quad A_3 e^{z} - B_3 e^{-z} - C_3(2-4\mu_3-z)e^{z} + D_3(2-4\mu_3+z)e^{-z} = 0$$

$$C_3 = 0$$

2. Resolution of the system of 10 boundary equations.

We write the conditions at the second interface in matrix form

$$M_3 (A_2\ B_2\ C_2\ D_2)^T = M_4 (B_3\ \ D_3)^T$$

and invert matrix M3

$$(A_2\ B_2\ C_2\ D_2)^T = M_3^{-1} \cdot M_4 (B_3\ \ D_3)^T$$

after having replaced A_3 by its value at the bottom

$$A_3 e^{y} = B_3 e^{-(2z-y)} + D_3(2-4\mu_3+z)e^{-(2z-y)}$$

$$M_3^{-1} = -\frac{1}{4(1-\mu_2)} e^{-y} \begin{pmatrix} -(1+y) & -(2-4\mu_2-y) & -(2\mu_2+y) & -(1-2\mu_2-y) \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-\frac{1}{4(1-\mu_2)} e^{y} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -(1-y) & (2-4\mu_2-y) & (2\mu_2-y) & -(1-2\mu_2+y) \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

$$M_4 = e^{-(2z-y)} \begin{pmatrix} 1 & 2-4\mu_3+z \\ 1 & 2-4\mu_3+z \\ k_w & k_w(2-4\mu_3+z) \\ k_u & k_u(2-4\mu_3+z) \end{pmatrix} + e^{-y} \begin{pmatrix} 1 & (1-2\mu_3+y) \\ -1 & (2\mu_3-y) \\ -k_w & -k_w(2-4\mu_3+y) \\ k_u & -k_u(1-y) \end{pmatrix}$$

$$(A_2\ B_2\ C_2\ D_2)^T = -\frac{1}{4(1-\mu_2)} \left[e^{-y}M_{31} + e^{y}M_{32}\right]\left[e^{-(2z-y)}M_{41} + e^{-y}M_{42}\right](B_3\ \ D_3)^T$$

$$= -\frac{1}{4(1-\mu_2)}\left[e^{-2z}M_{3141} + e^{-2y}M_{3142} + e^{-2(z-y)}M_{3241} + M_{3242}\right](B_3\ \ D_3)^T$$

$$M_{3141} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} \qquad M_{3142} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} \qquad M_{3241} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix}$$

$$M_{3242} = \begin{pmatrix} 0 & 0 \\ k_1 & k_4 \\ 0 & 0 \\ -k_3 & k_2 \end{pmatrix} + y \begin{pmatrix} 0 & 0 \\ k_3 & (k_1-k_2) \\ 0 & 0 \\ 0 & -k_3 \end{pmatrix} + y^2 \begin{pmatrix} 0 & 0 \\ 0 & k_3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ k_{21} & k_{22} \\ 0 & 0 \\ k_{41} & k_{42} \end{pmatrix}$$

$$K_1 = -3 + 4\mu_2 - K_n - 2\mu_2 (K_w - K_n)$$

$$K_2 = -1 - 2K_w + 4K_w \cdot \mu_3 - K_n$$

$$K_3 = K_w - K_n$$

$$K_4 = -1 + 6\mu_3 - 8\mu_2\mu_3 - 4\mu_2 K_w + 8\mu_2\mu_3 K_w + K_n - 2\mu_2 K_n$$

$$A_2 = -\frac{1}{4(1-\mu_2)} \left\{ \left[e^{-2(z-y)} M_{3141}(1,1) + M_{3142}(1,1) \right] B_3 e^{-2\lambda_1} + \left[e^{-2(z-y)} M_{3141}(1,2) + M_{3142}(1,2) \right] D_3 e^{-2\lambda_1} \right\}$$

$$= -\frac{1}{4(1-\mu_2)} \left[Q_{11} B_3 e^{-2\lambda_1} + Q_{12} D_3 e^{-2\lambda_1} \right]$$

$$B_2 = -\frac{1}{4(1-\mu_2)} \left\{ \left[e^{-2(z-y)} M_{3241}(2,1) + K_{21} \right] B_3 + \left[e^{-2(z-y)} M_{3241}(2,2) + K_{22} \right] D_3 \right\}$$

$$= -\frac{1}{4(1-\mu_2)} \left\{ \left[Q_{21} + K_{21} \right] B_3 + \left[Q_{22} + K_{22} \right] D_3 \right\}$$

$$C_2 = -\frac{1}{4(1-\mu_2)} \left\{ \left[e^{-2(z-y)} M_{3141}(3,1) + M_{3142}(3,1) \right] B_3 e^{-2\lambda_1} + \left[e^{2(z-y)} M_{3141}(3,2) + M_{3142}(3,2) \right] D_3 e^{-2\lambda_1} \right\}$$

$$= -\frac{1}{4(1-\mu_2)} \left[Q_{31} B_3 e^{-2\lambda_1} + Q_{32} D_3 e^{-2\lambda_1} \right]$$

$$D_2 = -\frac{1}{4(1-\mu_2)} \left\{ \left[e^{-2(z-y)} M_{3241}(4,1) + K_{41} \right] B_3 + \left[e^{-2(z-y)} M_{3241}(4,2) + K_{42} \right] D_3 \right\}$$

$$= -\frac{1}{4(1-\mu_2)} \left\{ \left[Q_{41} + K_{41} \right] B_3 + \left[Q_{42} + K_{42} \right] D_3 \right\}$$

$$(A_2\ B_2\ C_2\ D_2)^T = -\frac{1}{4(1-\mu_2)}\left[MQ + MK \right] (B_3\ D_3)^T$$

$$MQ = \begin{vmatrix} Q_{11}e^{-2\lambda_1} & Q_{12}e^{-2\lambda_1} \\ Q_{21} & Q_{22} \\ Q_{31}e^{-2\lambda_1} & Q_{32}e^{-2\lambda_1} \\ Q_{41} & Q_{42} \end{vmatrix} \qquad MK = \begin{vmatrix} 0 & 0 \\ K_{21} & K_{22} \\ 0 & 0 \\ K_{41} & K_{42} \end{vmatrix}$$

We write the boundary conditions at the first interface in matrix form

$$M_1(A_1\ B_1\ C_1\ D_1)^T = M_2\ (A_2\ B_2\ C_2\ D_2)^T$$

and invert matrix M1

$$(A_1\ B_1\ C_1\ D_1)^T = M_1^{-1}\ M_2\ (A_2\ B_2\ C_2\ D_2)^T$$

$M1^{-1}$ is given in § 1.2

$$M_1^{-1} = -\frac{1}{4(1-\mu_1)}\left[e^{-x} M_{11} + e^{x}\ M_{12} \right]$$

$$M_2 = e^{x}\begin{vmatrix} 1 & 0 & -(1-2\mu_2-x) & 0 \\ 1 & 0 & (2\mu_2+x) & 0 \\ F_w & 0 & -F_w(2-4\mu_2-x) & 0 \\ F_u & 0 & F_u(1+x) & 0 \end{vmatrix}$$

$$+\ e^{-x}\begin{vmatrix} 0 & 1 & 0 & (1-2\mu_2+x) \\ 0 & -1 & 0 & (2\mu_2-x) \\ 0 & -F_w & 0 & -F_w(2-4\mu_2+x) \\ 0 & F_u & 0 & -F_u(1-x) \end{vmatrix}$$

$$M_1^{-1}.M_2 = -\frac{1}{4(1-\mu_1)}\left[e^{-x} M_{11} + e^{x} M_{12} \right]\left[e^{x} M_{21} + e^{-x} M_{22} \right]$$

$$M_1^{-1} \cdot M_2 = -\frac{1}{4(1-\mu_1)} \left[M_{1121} + e^{-2k} M_{1122} + e^{2k} M_{1221} + M_{1222} \right]$$

$$M_{1122} = \begin{vmatrix} 0 & + & 0 & + \\ 0 & 0 & 0 & 0 \\ 0 & + & 0 & + \\ 0 & 0 & 0 & 0 \end{vmatrix} \qquad M_{1221} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ + & 0 & + & 0 \\ 0 & 0 & 0 & 0 \\ + & 0 & + & 0 \end{vmatrix}$$

$$M_{1121} + M_{1222} = \begin{vmatrix} F_{11} & 0 & F_{13} & 0 \\ 0 & F_{22} & 0 & F_{24} \\ F_{31} & 0 & F_{33} & 0 \\ 0 & F_{42} & 0 & F_{44} \end{vmatrix}$$

$$F_{11} = F_1 - x F_3$$

$$F_{13} = -F_4 + x(F_1 - F_2) - x^2 F_3$$

$$F_{22} = F_1 + x F_3$$

$$F_{24} = F_4 + x(F_1 - F_2) + x^2 F_3$$

$$F_{31} = F_3$$

$$F_{33} = F_2 + x F_3$$

$$F_{42} = -F_3$$

$$F_{44} = F_2 - x F_3$$

$$A_1 = -\frac{1}{4(1-\mu_1)} \left[F_{11} A_2 + M_{1122}(1,2) e^{-2k} B_2 + F_{13} C_2 + M_{1122}(1,4) e^{-2k} D_2 \right]$$

$$= -\frac{1}{4(1-\mu_1)} \left[F_{11} A_2 + Q_{12} e^{-2k} B_2 + F_{13} C_2 + Q_{14} e^{-2k} D_2 \right]$$

$$B_1 = -\frac{1}{4(1-\mu_1)}\left[M_{1221}(2,1)e^{2k}A_2 + F_{22}B_2 + M_{1221}(2,3)e^{2k}C_2 + F_{24}D_2\right]$$

$$= -\frac{1}{4(1-\mu_1)}\left[R_{21}e^{2k}A_2 + F_{22}B_2 + R_{23}e^{2k}C_2 + F_{24}D_2\right]$$

$$C_1 = -\frac{1}{4(1-\mu_1)}\left[F_{31}A_2 + M_{1122}(3,2)e^{-2k}B_2 + F_{33}C_2 + M_{1122}(3,4)e^{-2k}D_2\right]$$

$$= -\frac{1}{4(1-\mu_1)}\left[F_{31}A_2 + R_{32}e^{-2k}B_2 + F_{33}C_2 + R_{34}e^{-2k}D_2\right]$$

$$D_1 = -\frac{1}{4(1-\mu_1)}\left[M_{1221}(4,1)e^{2k}A_2 + F_{42}B_2 + M_{1221}(4,3)e^{2k}C_2 + F_{44}D_2\right]$$

$$= -\frac{1}{4(1-\mu_1)}\left[R_{41}e^{2k}A_2 + F_{42}B_2 + R_{43}e^{2k}C_2 + F_{44}D_2\right]$$

$$(A_1\ B_1\ C_1\ D_1)^T = -\frac{1}{4(1-\mu_1)}\left[MR + MF\right](A_2\ B_2\ C_2\ D_2)^T$$

$$MR = \begin{pmatrix} 0 & R_{12}e^{-2k} & 0 & R_{14}e^{-2k} \\ R_{21}e^{2k} & 0 & R_{23}e^{2k} & 0 \\ 0 & R_{32}e^{-2k} & 0 & R_{34}e^{-2k} \\ R_{41}e^{2k} & 0 & R_{43}e^{2k} & 0 \end{pmatrix}$$

We write the boundary conditions at the surface in matrix form

$$M_0\,(A_1\ B_1\ C_1\ D_1)^T = (1 \quad 0)^T$$

$$M_0\left[MR + MF\right]\left[MQ + Mk\right](B_3 \quad D_3)^T = (16(1-\mu_1)(1-\mu_2) \qquad 0)^T$$

$$MR.MQ + MR.Mk + MF.MQ = \begin{vmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \\ B_{41} & B_{42} \end{vmatrix} = MB$$

$$B_{11} = \left[R_{12}(Q_{21}+k_{21}) + R_{14}(Q_{41}+k_{41})\right]e^{-x} + (Q_{11}F_{11} + F_{13}Q_{31})e^{-2y}$$

$$B_{12} = \left[R_{12}(Q_{22}+k_{22}) + R_{14}(Q_{42}+k_{42})\right]e^{-x} + (F_{11}Q_{12} + F_{13}Q_{32})e^{-2y}$$

$$B_{21} = \left[R_{21}.Q_{11} + R_{23}.Q_{31}\right]e^{-2(y-x)} + \left[F_{22}.Q_{21} + F_{24}.Q_{41}\right]$$

$$B_{22} = \left[R_{21}.Q_{12} + R_{23}.Q_{32}\right]e^{-2(y-x)} + \left[F_{22}.Q_{22} + F_{24}.Q_{42}\right]$$

$$B_{31} = \left[R_{32}(Q_{21}+k_{21}) + R_{34}(Q_{41}+k_{41})\right]e^{-x} + (F_{31}Q_{11} + F_{33}Q_{31})e^{-2y}$$

$$B_{32} = \left[R_{32}(Q_{22}+k_{22}) + R_{34}(Q_{42}+k_{42})\right]e^{-x} + (F_{31}Q_{12} + F_{33}Q_{32})e^{-2y}$$

$$B_{41} = \left[R_{41}.Q_{11} + R_{43}.Q_{31}\right]e^{-2(y-x)} + \left[F_{42}.Q_{21} + F_{44}.Q_{41}\right]$$

$$B_{42} = \left[R_{41}.Q_{12} + R_{43}.Q_{32}\right]e^{-2(y-x)} + \left[F_{42}.Q_{22} + F_{44}.Q_{42}\right]$$

$$MF.Mk = \begin{vmatrix} 0 & 0 \\ Fk_{21} & Fk_{22} \\ 0 & 0 \\ Fk_{41} & Fk_{42} \end{vmatrix} = MFk$$

$$M_o.MB = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = MA$$

$$A_{11} = B_{11} + B_{21} - (1-2\mu_1)B_{31} + (1-2\mu_1)B_{41}$$

$$A_{12} = B_{12} + B_{22} - (1-2\mu_1)B_{32} + (1-2\mu_1)B_{42}$$

$$A_{21} = B_{11} - B_{21} + 2\mu_1 B_{31} + 2\mu_1 B_{41}$$

$$A_{22} = B_{12} - B_{22} + 2\mu_1 B_{32} + 2\mu_1 B_{42}$$

$$M_o . MFk = \begin{vmatrix} FF_{11} & FF_{12} \\ FF_{21} & FF_{22} \end{vmatrix} = MFF$$

$$FF_{11} = \quad Fk_{21} + (1-2\mu_1)\, Fk_{41}$$

$$FF_{12} = \quad Fk_{22} + (1-2\mu_1)\, Fk_{42}$$

$$FF_{21} = -Fk_{21} + 2\mu_1\, Fk_{41}$$

$$FF_{22} = -Fk_{22} + 2\mu_1\, Fk_{42}$$

$$(MA + MFF)(B_3 \quad D_3)^T = (16(1-\mu_1)(1-\mu_2) \quad 0)^T$$

We develop the matrix equation

$$(A11 + FF11)\, B_3 + (A12 + FF12)\, D_3 = 16(1-\mu_1)(1-\mu_2)$$

$$(A21 + FF21)\, B_3 + (A22 + FF22)\, D3 = 0$$

and solve the system

$$B_3 = 16(1-\mu_1)(1-\mu_2)\ \frac{A22 + FF22}{\nabla}$$

$$D_3 = -16(1-\mu_1)(1-\mu_2)\ \frac{A21 + FF21}{\nabla}$$

$$\nabla = A_{11}(A22 + FF22) - A12\,(A21 + FF21) + FF11.A22$$

$$-FF12.A21 + FF11.FF22 - FF12.FF21$$

$$FF11.FF22 - FF12.FF21 = (k_1.k_2 + k_3.k_4)(F_1F_2 + F_3F_4)$$

3. Relations for the parameters.

These relations are deduced from the different matrix equations

$$A_2 e^{Y} = - \frac{1}{4(1-\mu_2)} \left[Q_{11} \cdot B_3 e^{-Y} + Q_{12} \cdot D_3 e^{-Y} \right]$$

$$B_2 e^{-X} = - \frac{1}{4(1-\mu_2)} \left[(Q_{21} + k_{21}) B_3 e^{-X} + (Q_{22} + k_{22}) D_3 e^{-X} \right]$$

$$C_2 e^{Y} = - \frac{1}{4(1-\mu_2)} \left[Q_{31} \cdot B_3 e^{-Y} + Q_{32} \cdot D_3 e^{-Y} \right]$$

$$D_2 e^{-Y} = - \frac{1}{4(1-\mu_2)} \left[(Q_{41} + k_{41}) B_3 e^{-X} + (Q_{42} + k_{42}) D_3 e^{-X} \right]$$

$$A_1 e^{X} = - \frac{1}{4(1-\mu_1)} \left[F_{11} (A_2 e^{Y}) e^{-(Y-X)} + R_{12} (B_2 e^{-Y}) + F_{13} (C_2 e^{Y}) e^{-(Y-X)} + F_{14} (D_2 e^{-X}) \right]$$

$$C_1 e^{X} = - \frac{1}{4(1-\mu_1)} \left[P_{31} (A_2 e^{Y}) e^{-(Y-X)} + R_{32} (B_2 e^{-X}) + F_{33} (C_2 e^{Y}) e^{-(Y-X)} + F_{34} (D_2 e^{-X}) \right]$$

$$B_3 e^{-Y} = \frac{16(1-\mu_1)(1-\mu_2)(A_{22} + FF_{22}) e^{-Y}}{\nabla}$$

$$D_3 e^{-Y} = - \frac{16(1-\mu_1)(1-\mu_2)(A_{21} + FF_{21}) e^{-Y}}{\nabla}$$

Chapter 3. The four layered structure.

1. Supplementary boundary conditions.

Boundary conditions at the third interface ($z = H_1 + H_2 + H_3$)

$$\sigma_z: \quad A_3 e^{z} + B_3 e^{-z} - C_3(1-2\mu_3 - z)e^{z} + D_3(1-2\mu_3+z)e^{-z} =$$
$$A_4 e^{z} + B_4 e^{-z} - C_4(1-2\mu_4 - z)e^{z} + D_4(1-2\mu_4+z)e^{-z}$$

$$\tau_{rz}: \quad A_3 e^{z} - B_3 e^{-z} + C_3(2\mu_3 + z)e^{z} + D_3(2\mu_3 - z)e^{-z} =$$
$$A_4 e^{z} - B_4 e^{-z} + C_4(2\mu_4 + z)e^{z} + D_4(2\mu_4 - z)e^{-z}$$

$$w: \quad A_3 e^{z} - B_3 e^{-z} - C_3(2-4\mu_3 - z)e^{z} - D_3(2-4\mu_3+z)e^{-z} =$$
$$L_w\left[A_4 e^{z} - B_4 e^{-z} - C_4(2-4\mu_4 - z)e^{z} - D_4(2-4\mu_4+z)e^{-z}\right]$$

$$u: \quad A_3 e^{z} + B_3 e^{-z} + C_3(1+z)e^{z} - D_3(1-z)e^{-z} =$$
$$L_u\left[A_4 e^{z} + B_4 e^{-z} + C_4(1+z)e^{z} - D_4(1-z)e^{-z}\right]$$

Boundary conditions at the bottom ($z = H_1 + H_2 + H_3 + H_4$)

$$w: \quad A_4 e^{t} - B_4 e^{-t} - C_4(2-4\mu_4 - t)e^{t} - D_4(2-4\mu_4+t)e^{-t} = 0$$

$$C_4 = 0$$

2. Resolution of the system of 14 boundary equations.

We write the conditions at the third interface in matrix form

$$M_5 \, (A_3 \; B_3 \; C_3 \; D_3)^T = M_6 \, (B_4 \;\; D_4)^T$$

and invert matrix M5

$$(A_3 \; B_3 \; C_3 \; D_3)^T = M_5^{-1} \cdot M_6 \, (B_4 \;\; D_4)^T$$

after having replaced A_4 by its value at the bottom

$$A_4 e^{z} = B_4 e^{-(2t-z)} + D_4(2-4\mu_4+t)e^{-(2t-z)}$$

$$M_5^{-1} = -\frac{1}{4(1-\mu_3)} e^{-z} \begin{pmatrix} -(1+z) & -(2-4\mu_3-z) & -(2\mu_3+z) & -(1-2\mu_3-z) \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-\frac{1}{4(1-\mu_3)} e^{z} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -(1-z) & (2-4\mu_3-z) & (2\mu_3-z) & -(1-2\mu_3+z) \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

$$M_6 = e^{-(2t-z)} \begin{pmatrix} 1 & 2-4\mu_4+t \\ 1 & 2-4\mu_4+t \\ Lw & Lw(2-4\mu_4+t) \\ Lu & Lu(2-4\mu_4+t) \end{pmatrix} + e^{-z} \begin{pmatrix} 1 & 1-2\mu_4+z \\ -1 & 2\mu_4-z \\ -Lw & -Lw(2-4\mu_4+z) \\ Lu & -Lu(1+z) \end{pmatrix}$$

$$(A_3\, B_3\, C_3\, D_3)^T = -\frac{1}{4(1-\mu_3)} \left[e^{-z} M_{51} + e^{z} M_{52}\right]\left[e^{-(2t-z)} M_{61} + e^{-z} M_{62}\right] (B_4 \;\; D_4)^T$$

$$= -\frac{1}{4(1-\mu_3)} \left[e^{-2t} M_{5161} + e^{-2z} M_{5162} + e^{-2(t-z)} M_{5261} + M_{5262}\right] (B_4 \;\; D_4)^T$$

$$M_{5161} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} \qquad M_{5162} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} \qquad M_{5261} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix}$$

$$M_{5262} = \begin{pmatrix} 0 & 0 \\ L_1 & L_4 \\ 0 & 0 \\ -L_3 & L_2 \end{pmatrix} + z \begin{pmatrix} 0 & 0 \\ L_3 & L_1-L_2 \\ 0 & 0 \\ 0 & -L_3 \end{pmatrix} + z^2 \begin{pmatrix} 0 & 0 \\ 0 & L_3 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ L_{21} & L_{22} \\ 0 & 0 \\ L_{41} & L_{42} \end{pmatrix}$$

$$L_1 = -3 + 4\mu_3 - L_u - 2\mu_3 (L_w - L_u)$$

$$L_2 = -1 - 2L_w + 4L_w \cdot \mu_4 - L_u$$

$$L_3 = L_w - L_u$$

$$L_4 = -1 + 6\mu_4 - 8\mu_3\mu_4 - 4\mu_3 L_w + 8\mu_3\mu_4 L_w + L_u - 2\mu_3 L_u$$

$$A_3 = -\frac{1}{4(1-\mu_3)} \left\{ \left[e^{-2(t-x)} M_{5161}(1,1) + M_{5162}(1,1) \right] B_4 e^{-2x} + \left[e^{-2(t-x)} M_{5161}(1,2) + M_{5162}(1,2) \right] D_4 e^{-2x} \right\}$$

$$= -\frac{1}{4(1-\mu_3)} \left[P_{11} B_4 e^{-2x} + P_{12} D_4 e^{-2x} \right]$$

$$B_3 = -\frac{1}{4(1-\mu_3)} \left\{ \left[e^{-2(t-x)} M_{5261}(2,1) + M_{5262}(2,1) \right] B_4 + \left[e^{-2(t-x)} M_{5261}(2,2) + M_{5262}(2,2) \right] D_4 \right\}$$

$$= -\frac{1}{4(1-\mu_3)} \left\{ \left[P_{21} + L_{21} \right] B_4 + \left[P_{22} + L_{22} \right] D_4 \right\}$$

$$C_3 = -\frac{1}{4(1-\mu_3)} \left\{ \left[e^{-2(t-x)} M_{5161}(3,1) + M_{5162}(3,1) \right] B_4 e^{-2x} + \left[e^{-2(t-x)} M_{5161}(3,2) + M_{5162}(3,2) \right] D_4 e^{-2x} \right\}$$

$$= -\frac{1}{4(1-\mu_3)} \left[P_{31} B_4 e^{-x} + P_{32} D_4 e^{-x} \right]$$

$$D_3 = -\frac{1}{4(1-\mu_3)} \left\{ \left[e^{-2(t-x)} M_{5261}(4,1) + M_{5262}(4,1) \right] B_4 + \left[e^{-2(t-x)} M_{5261}(4,2) + M_{5262}(4,2) \right] D_4 \right\}$$

$$= -\frac{1}{4(1-\mu_3)} \left\{ \left[P_{41} + L_{41} \right] B_4 + \left[P_{42} + L_{42} \right] D_4 \right\}$$

$$(A_3 \ B_3 \ C_3 \ D_3)^T = -\frac{1}{4(1-\mu_3)} \left[MP + ML \right] (B_4 \ \ D_4)^T$$

$$MP = \begin{vmatrix} P_{11}e^{-2x} & P_{12}e^{-2x} \\ P_{21} & P_{22} \\ P \; e^{-2x} & P_{32}e^{-2x} \\ P_{41} & P_{42} \end{vmatrix} \qquad ML = \begin{vmatrix} 0 & 0 \\ L_{21} & L_{22} \\ 0 & 0 \\ L_{41} & L_{42} \end{vmatrix}$$

We write the boundary conditions at the second interface in matrix form

$$M_3 \, (A_2 B_2 C_2 D_2)^T = M_4 \, (A_3 \; B_3 \; C_3 \; D_3)^T$$

and invert matrix M3

$$(A_2 B_2 C_2 D_2)^T = M_3^{-1} . M_4 \, (A_3 B_3 C_3 D_3)^T$$

$M3^{-1}$ is given in § 2.2

$$M_3^{-1} = - \frac{1}{4(1-\mu_2)} \left[e^{-y} M_{31} + e^{y} M_{32} \right]$$

$$M_4 = e^{y} \begin{vmatrix} 1 & 0 & -(1-2\mu_3-y) & 0 \\ 1 & 0 & 2\mu_3+y & 0 \\ kw & 0 & -kw(2-4\mu_3-y) & 0 \\ ku & 0 & ku(1+y) & 0 \end{vmatrix}$$

$$+ e^{-y} \begin{vmatrix} 0 & 1 & 0 & (1-2\mu_3+y) \\ 0 & -1 & 0 & (2\mu_3-y) \\ 0 & -kw & 0 & -kw(2-4\mu_3+y) \\ 0 & ku & 0 & -ku(1-y) \end{vmatrix}$$

$$M_3^{-1} . M_4 = - \frac{1}{4(1-\mu_2)} \left[e^{-y} M_{31} + e^{y} M_{32} \right] \left[e^{y} M_{41} + e^{-y} M_{42} \right]$$

$$= - \frac{1}{4(1-\mu_2)} \left[M_{3141} + e^{-2y} M_{3142} + e^{2y} M_{3241} + M_{3242} \right]$$

$$M_{3142} = \begin{pmatrix} 0 & Q_{12} & 0 & Q_{14} \\ 0 & 0 & 0 & 0 \\ 0 & Q_{32} & 0 & Q_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_{3241} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ Q_{21} & 0 & Q_{23} & 0 \\ 0 & 0 & 0 & 0 \\ Q_{41} & 0 & Q_{43} & 0 \end{pmatrix}$$

$$M_{3141} + M_{3242} = \begin{pmatrix} k_1 - y k_3 & 0 & -k_4 + y(k_1 - k_3) - y^2 k_3 & 0 \\ 0 & k_1 + y k_3 & 0 & k_4 + y(k_1 - k_2) + y^2 k_3 \\ k_3 & 0 & k_2 + y k_3 & 0 \\ 0 & -k_3 & 0 & k_2 - y k_3 \end{pmatrix}$$

$$= \begin{pmatrix} k_{11} & 0 & k_{13} & 0 \\ 0 & k_{22} & 0 & k_{24} \\ k_{31} & 0 & k_{33} & 0 \\ 0 & k_{42} & 0 & k_{44} \end{pmatrix}$$

$$A_2 = -\frac{1}{4(1-\mu_2)} \left[k_{11} A_3 + Q_{12} B_3 e^{-2y} + k_{13} C_3 + Q_{14} D_3 e^{-2y} \right]$$

$$B_2 = -\frac{1}{4(1-\mu_2)} \left[Q_{21} A_3 e^{2y} + k_{22} B_3 + Q_{23} C_3 e^{2y} + k_{24} D_3 \right]$$

$$C_2 = -\frac{1}{4(1-\mu_2)} \left[k_{31} A_3 + Q_{32} B_3 e^{-2y} + k_{33} C_3 + Q_{34} D_3 e^{-2y} \right]$$

$$D_2 = -\frac{1}{4(1-\mu_2)} \left[Q_{41} A_3 e^{2y} + k_{42} B_3 + Q_{43} C_3 e^{2y} + k_{44} D_3 \right]$$

The relation for $(A_1\ B_1\ C_1\ D_1)^T$ is given in § 2.2

$$(A_1\ B_1\ C_1\ D_1)^T = -\frac{1}{4(1-\mu_1)} \left[MR + MF \right] (A_2\ B_2\ C_2\ D_2)^T$$

$$M_0 \left[MR + MF \right] \left[MQ + MK \right] \left[MP + ML \right] = \left(-64(1-\mu_1)(1-\mu_2)(1-\mu_3) \quad 0 \right)^T$$

$M_0 . MR =$

$$\begin{vmatrix} 1 & 1 & -(1-2\mu_1) & (1+2\mu_1) \\ 1 & -1 & 2\mu_1 & 2\mu_1 \end{vmatrix} \begin{vmatrix} 0 & R_{12}e^{-2x} & 0 & R_{14}e^{-2x} \\ R_{21}e^{2x} & 0 & R_{23}e^{2x} & 0 \\ 0 & R_{32}e^{-2x} & 0 & R_{34}e^{-2x} \\ R_{41}e^{2x} & 0 & R_{43}e^{2x} & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{vmatrix} .$$

$$a_{11} = e^{2x}\left[R_{21} + (1-2\mu_1) R_{41}\right] = e^{2x} . a'_{11}$$

$$a_{12} = e^{-2x}\left[R_{12} - (1-2\mu_1) R_{32}\right] = e^{-2x} . a'_{12}$$

$$a_{13} = e^{2x}\left[R_{23} + (1-2\mu_1) R_{43}\right] = e^{2x} . a'_{13}$$

$$a_{14} = e^{-2x}\left[R_{14} - (1-2\mu_1) R_{34}\right] = e^{-2x} . a'_{14}$$

$$a_{21} = e^{2x}\left[-R_{21} + 2\mu_1 R_{41}\right] = e^{2x} a'_{21}$$

$$a_{22} = e^{-2x}\left[R_{12} + 2\mu_1 R_{32}\right] = e^{-2x} a'_{22}$$

$$a_{23} = e^{2x}\left[-R_{23} + 2\mu_1 R_{43}\right] = e^{2x} a'_{23}$$

$$a_{24} = e^{-2x}\left[R_{14} + 2\mu_1 R_{34}\right] = e^{-2x} a'_{24}$$

$M_0 . M_F =$

$$\begin{vmatrix} 1 & 1 & -(1-2\mu_1) & (1-2\mu_1) \\ 1 & -1 & 2\mu_1 & 2\mu_1 \end{vmatrix} \begin{vmatrix} F_{11} & 0 & F_{13} & 0 \\ 0 & F_{22} & 0 & F_{24} \\ F_{31} & 0 & F_{33} & 0 \\ 0 & F_{42} & 0 & F_{44} \end{vmatrix}$$

$$= \begin{vmatrix} FF_{11} & FF_{12} & FF_{13} & FF_{14} \\ FF_{21} & FF_{22} & FF_{23} & FF_{24} \end{vmatrix}$$

$$FF_{11} = F_{11} - (1-2\mu_1) F_{31}$$
$$FF_{12} = F_{22} + (1-2\mu_1) F_{42}$$
$$FF_{13} = F_{13} - (1-2\mu_1) F_{33}$$
$$FF_{14} = F_{24} + (1-2\mu_1) F_{44}$$
$$FF_{21} = F_{11} + 2\mu_1 F_{31}$$
$$FF_{22} = -F_{22} + 2\mu_1 F_{42}$$
$$FF_{23} = F_{13} + 2\mu_1 F_{33}$$
$$FF_{24} = -F_{24} + 2\mu_1 F_{44}$$

$$[MQ + MK][MP + ML] = [MQ.MP + MQ.ML + MK.MP] + MK.ML$$

$$MQ.MP + MQ.ML + MK.MP = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{vmatrix}$$

$$b_{11} = e^{-2y}\left[(k_{11}.P_{11} + k_{13}P_{31})\,e^{-2(z-y)} + (Q_{12}.P_{21} + Q_{14}.P_{41}) + (Q_{12}L_{21} + Q_{14}.L_{41})\right] = e^{-2y}\,b'_{11}$$
$$b_{12} = e^{-2y}\left[(k_{11}.P_{12} + k_{13}P_{32})\,e^{-2(z-y)} + (Q_{12}P_{22} + Q_{14}P_{42}) + (Q_{12}L_{22} + Q_{14}L_{42})\right] = e^{-2y}\,b'_{12}$$

$$b_{21} = (k_{22}P_{21} + k_{24}P_{41}) + (Q_{21}P_{11} + Q_{23}P_{31})\,e^{-2(z-y)} = b'_{21}$$
$$b_{22} = (k_{22}P_{22} + k_{24}P_{42}) + (Q_{21}P_{12} + Q_{23}P_{32})\,e^{-2(z-y)} = b'_{22}$$
$$b_{31} = e^{-2y}\left[(k_{31}P_{11} + k_{33}P_{31})\,e^{-2(z-y)} + (Q_{32}P_{21} + Q_{34}P_{41}) + (Q_{32}L_{21} + Q_{34}L_{41})\right] = e^{-2y}\,b'_{31}$$
$$b_{32} = e^{-2y}\left[(k_{31}P_{12} + k_{33}P_{32})\,e^{-2(z-y)} + (Q_{32}P_{22} + Q_{34}P_{42}) + (Q_{32}L_{22} + Q_{34}L_{42})\right] = e^{-2y}\,b'_{32}$$
$$b_{41} = (k_{42}P_{21} + k_{44}P_{41}) + (Q_{41}P_{11} + Q_{43}P_{31})\,e^{-2(z-y)} = b'_{41}$$
$$b_{42} = (k_{42}P_{22} + k_{44}P_{42}) + (Q_{42}P_{12} + Q_{43}P_{32})\,e^{-2(z-y)} = b'_{42}$$

$Mk . ML =$

$$\begin{vmatrix} k_{11} & 0 & k_{13} & 0 \\ 0 & k_{22} & 0 & k_{24} \\ k_{31} & 0 & k_{33} & 0 \\ 0 & k_{42} & 0 & k_{44} \end{vmatrix} \begin{vmatrix} 0 & 0 \\ L_{21} & L_{22} \\ 0 & 0 \\ L_{41} & L_{42} \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ kL_{21} & kL_{22} \\ 0 & 0 \\ kL_{41} & kL_{24} \end{vmatrix}$$

$$kL_{21} = k_{22} . L_{21} + k_{24} . L_{41}$$

$$kL_{22} = k_{22} . L_{22} + k_{24} . L_{42}$$

$$kL_{41} = k_{42} . L_{21} + k_{44} . L_{41}$$

$$kL_{42} = k_{42} . L_{22} + k_{44} . L_{42}$$

$$(MA + MFF) . (MB + MkL) = [MA.MB + MA.MkL + MFF.MB] + MFF.MkL$$

$$[MA.MB + MA.MKL + MFF.MB] = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}$$

$$c_{11} = a'_{11} b'_{11} e^{-2(y-x)} + a'_{12} b'_{21} e^{-2x} + a'_{13} b'_{31} e^{-2(y-x)} + a'_{14} b'_{41} e^{-2x}$$
$$+ ff_{11} b'_{11} e^{-2y} + ff_{12} b'_{21} + ff_{13} b'_{31} e^{-2y} + ff_{14} b'_{41}$$
$$+ [a'_{12} kL_{21} + a'_{14} kL_{41}] e^{-2x}$$

$$c_{12} = a'_{11} b'_{12} e^{-2(y-x)} + a'_{12} b'_{22} e^{-2x} + a'_{13} b'_{32} e^{-2(y-x)} + a'_{14} b'_{42} e^{-2x}$$
$$+ ff_{11} b'_{12} e^{-2y} + ff_{12} b'_{22} + ff_{13} b'_{32} e^{-2y} + ff_{14} b'_{42}$$
$$+ [a'_{12} kL_{22} + a'_{14} kL_{42}] e^{-2x}$$

$$c_{21} = a'_{21} b'_{11} e^{-2(y-x)} + a'_{22} b'_{21} e^{-2x} + a'_{23} b'_{31} e^{-2(y-x)} + a'_{24} b'_{41} e^{-2x}$$
$$+ ff'_{21} b'_{11} e^{-2y} + ff'_{22} \cdot b'_{21} + ff'_{23} b'_{31} e^{-2y} + ff'_{24} b'_{41}$$
$$+ \left[a'_{22} kL_{21} + a'_{24} kL_{41} \right] e^{-2x}$$

$$c_{22} = a'_{21} b'_{12} e^{-2(y-x)} + a'_{22} b'_{22} e^{-2x} + a'_{23} b'_{32} e^{-2(y-x)} + a'_{24} b'_{42}$$
$$+ ff'_{21} b_{12} e^{-2y} + ff'_{22} \cdot b'_{22} + ff'_{23} b'_{32} e^{-2y} + ff'_{24} b'_{42}$$
$$+ \left[a'_{22} kL_{22} + a'_{24} kL_{42} \right] e^{-2x}$$

$$MFF.MKL = \begin{vmatrix} FkL_{11} & FkL_{12} \\ FkL_{21} & FkL_{22} \end{vmatrix}$$

$$FkL_{11} = FF_{12} . kL_{21} + FF_{14} . kL_{41}$$
$$FkL_{12} = FF_{12} . kL_{22} + FF_{14} . kL_{42}$$
$$FkL_{21} = FF_{22} . kL_{21} + FF_{24} . kL_{41}$$
$$FkL_{22} = FF_{22} . kL_{22} + FF_{24} . kL_{42} .$$

$$\left[\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} + \begin{pmatrix} FkL_{11} & FkL_{12} \\ FkL_{21} & FkL_{22} \end{pmatrix} \right] \begin{pmatrix} B_4 \\ D_4 \end{pmatrix} = \begin{pmatrix} -64(1-\mu_1)(1-\mu_2)(1-\mu_3) \\ 0 \end{pmatrix}$$

We develop the matrix equation

$$(c_{11} + FkL_{11}) B_4 + (c_{12} + FkL_{12}) D_4 = -64(1-\mu_1)(1-\mu_2)(1-\mu_3)$$

$$(c_{21} + FkL_{21}) B_4 + (c_{22} + FkL_{22}) D_4 = 0 .$$

$$B_4 = -64(1-\mu_1)(1-\mu_2)(1-\mu_3) \frac{c_{22} + FkL_{22}}{\nabla}$$

$$D_4 = 64(1-\mu_1)(1-\mu_2)(1-\mu_3) \frac{c_{21} + FkL_{21}}{\nabla}$$

$$\nabla = c_{11}\left(c_{22} + FkL_{22}\right) - c_{21}\left(c_{12} + FkL_{12}\right)$$
$$+ FkL_{11}.c_{22} - FkL_{21}.c_{12} + FkL_{11}.FkL_{22} - FkL_{21}.FkL_{12}.$$

One verifies that the product

$$FkL_{11}.FkL_{22} - FkL_{21}.FkL_{12} =$$
$$(k_1k_2 + k_3k_4)(L_1L_2 + L_3L_4)(F_1F_2 + F_3F_4)$$

is a constant.

3. Relations for the parameters.

These relations are deduced from the different matrix equations

$$A_3e^{z} = -\frac{1}{4(1-\mu_3)}\cdot\left[P_{11}.B_4e^{-z} + P_{12}.D_4e^{-z}\right]$$

$$B_3e^{-y} = -\frac{1}{4(1-\mu_3)}\left[(P_{21}+L_{21})B_4e^{-y} + (P_{22}+L_{22})D_4e^{-y}\right]$$

$$C_3e^{z} = -\frac{1}{4(1-\mu_3)}\left[P_{31}.B_4e^{-z} + P_{32}.D_4e^{-z}\right]$$

$$D_3e^{-y} = -\frac{1}{4(1-\mu_3)}\left[(P_{41}+L_{41})B_4e^{-y} + (P_{42}+L_{42})D_4e^{-y}\right]$$

$$A_2e^{y} = -\frac{1}{4(1-\mu_2)}\left[K_{11}(A_3e^{z})e^{-(z-y)} + Q_{12}(B_3e^{-y}) + K_{13}(C_3e^{z})e^{-(z-y)} + Q_{14}(D_3e^{-y})\right]$$

$$B_2e^{-y} = -\frac{1}{4(1-\mu_2)}\left[Q_{21}(A_3e^{z})e^{-(z-y)} + K_{22}(B_3e^{-y}) + Q_{23}(C_3e^{z})e^{-(z-y)} + K_{24}(D_3e^{-y})\right]$$

$$C_2e^{y} = -\frac{1}{4(1-\mu_2)}\left[K_{31}(A_3e^{z})e^{-(z-y)} + Q_{32}(B_3e^{-y}) + K_{33}(C_3e^{z})e^{-(z-y)} + Q_{34}(D_3e^{-y})\right]$$

$$D_2e^{-y} = -\frac{1}{4(1-\mu_2)}\left[Q_{41}(A_3e^{z})e^{-(z-y)} + K_{42}(B_3e^{-y}) + Q_{43}(C_3e^{z})e^{-(z-y)} + K_{44}(D_3e^{-y})\right]$$

The computation of the parameters B_2e^{-x} and D_2e^{-x} needs that of the intermediate parameters A_3e^{2y}, B_3^{-x}, C_3e^{2y} and D_3e^{-x}:

$$A_3e^{2y} = -\frac{1}{4(1-\mu_3)}\left[P_{11}\ B_4e^{-2(z-y)} + P_{12}\ D_4e^{-2(z-y)}\right]$$

$$B_3e^{-x} = -\frac{1}{4(1-\mu_3)}\left[(P_{21}+L_{21})\ B_4e^{-x} + (P_{22}+L_{22})\ D_4e^{-x}\right]$$

$$C_3e^{2y} = -\frac{1}{4(1-\mu_3)}\left[P_{31}\ B_4e^{-2(z-y)} + P_{32}\ D_4e^{-2(z-y)}\right]$$

$$D_3e^{-x} = -\frac{1}{4(1-\mu_3)}\left[(P_{41}+L_{41})\ B_4e^{-x} + (P_{42}+L_{42})\ D_4\ e^{-x}\right]$$

$$B_2e^{-x} = -\frac{1}{4(1-\mu_2)}\left[Q_{21}\,(A_3e^{2y})e^{-x} + K_{21}\,(B_3e^{-x}) + Q_{23}\,(C_3e^{2y})e^{-x} + K_{24}\,D_3e^{-x}\right]$$

$$D_2e^{-x} = -\frac{1}{4(1-\mu_2)}\left[Q_{41}\,(A_3e^{2y})e^{-x} + K_{42}\,(B_3e^{-x}) + Q_{43}\,(C_3e^{2y})e^{-x} + K_{44}\,D_3e^{-x}\right]$$

$$A_1e^{x} = -\frac{1}{4(1-\mu_1)}\left[F_{11}\,(A_2e^{y})e^{-(y-x)} + R_{12}\,(B_2e^{-x}) + F_{13}\,(C_2e^{y})e^{-(y-x)} + R_{14}\,(D_2e^{-x})\right]$$

$$C_1e^{x} = -\frac{1}{4(1-\mu_1)}\left[F_{31}\,(A_2e^{y})e^{-(y-x)} + R_{32}\,(B_2e^{-x}) + F_{33}\,(C_2e^{y})e^{-(y-x)} + R_{34}\,(D_2e^{-x})\right]$$

$$B_4e^{-z} = -64\,(1-\mu_1)(1-\mu_2)(1-\mu_3)\ \frac{c_{22}+FKL_{22}}{\nabla}\,e^{-z}$$

$$D_4e^{-z} = 64\,(1-\mu_1)(1-\mu_2)(1-\mu_3)\ \frac{c_{21}+FKL_{21}}{\nabla}\,e^{-z}$$

4. Determination of the stresses and the displacements.

4.1. Mathematical procedure. (in the case of a flexible load)

The program calculates the stresses and the displacements using following relations:

$$\sigma_z = pa\int_0^\infty J_0(mr).J_1(ma)\Big[A_i m^2 e^{mz} + B_i m^2 e^{-mz} - C_i m(1-2\mu_i - mz)e^{mz} + D_i m(1-2\mu_i + mz)e^{-mz}\Big]dm$$

$$\frac{\sigma_r+\sigma_\theta}{2} = -\frac{pa}{2}\int_0^\infty J_0(mr).J_1(ma)\Big[A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i m(1+4\mu_i + mz)e^{mz} - D_i m(1+4\mu_i - mz)e^{-mz}\Big]dm$$

$$\frac{\sigma_r-\sigma_\theta}{2} = -\frac{pa}{2}\int_0^\infty J_0(mr).J_1(ma)\Big[A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i m(1+mz)e^{mz} - D_i m(1-mz)e^{-mz}\Big]dm$$
$$+ pa\int_0^\infty \frac{J_1(mr).J_1(ma)}{mr}\Big[A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i m(1+mz)e^{mz} - D_i m(1-mz)e^{-mz}\Big]dm$$

$$\tau_{rz} = -pa\int_0^\infty J_1(mr).J_1(ma)\Big[A_i m^2 e^{mz} - B_i m^2 e^{-mz} + C_i m(2\mu_i + mz)e^{mz} + D_i m(2\mu_i - mz)e^{-mz}\Big]dm$$

$$w = +\frac{1+\mu_i}{E_i}.pa\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}\Big[A_i m^2 e^{mz} - B_i m^2 e^{-mz} - C_i m(2-4\mu_i - mz)e^{mz} - D_i m(2-4\mu_i + mz)e^{-mz}\Big]dm$$

$$u = -\frac{1+\mu_i}{E_i}.pa\int_0^\infty \frac{J_1(mr).J_1(ma)}{m}\Big[A_i m^2 e^{mz} + B_i m^2 e^{-mz} + C_i m(1+mz)e^{mz} - D_i m(1-mz)e^{-mz}\Big]dm$$

Stresses and displacements are calculated in a system of cylindrical coordinates. The stresses due to several loads are to be added together. Therefore we must express them in cartesian coordinates using following relations:

$$\sigma_x = \sigma_r \cdot \cos^2\alpha + \sigma_\theta \cdot \sin^2\alpha = \frac{\sigma_r + \sigma_\theta}{2} + \frac{\sigma_r - \sigma_\theta}{2} \cos 2\alpha$$

$$\sigma_y = \sigma_r \cdot \sin^2\alpha + \sigma_\theta \cdot \cos^2\alpha = \frac{\sigma_r + \sigma_\theta}{2} - \frac{\sigma_r - \sigma_\theta}{2} \cos 2\alpha$$

$$\sigma_z = \sigma_z$$

$$\tau_{yz} = \tau_{rz} \cdot \sin\alpha$$

$$\tau_{xz} = \tau_{rz} \cdot \cos\alpha$$

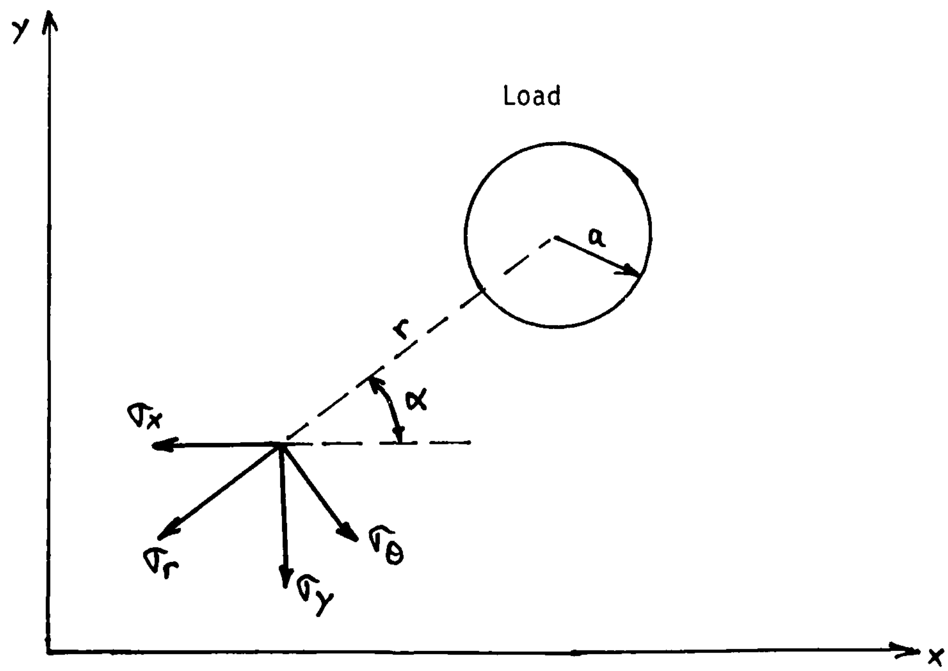
$$\tau_{xy} = (\sigma_r - \sigma_\theta) \cos\alpha \cdot \sin\alpha = \frac{\sigma_r - \sigma_\theta}{2} \cdot \sin 2\alpha$$

$$u_x = u_r \cdot \cos\alpha$$

$$v_y = u_r \cdot \sin\alpha$$

$$w = w$$

wherein the signification of the angle α is illustrated below.



4.2. Stresses and displacements at the surface (z = 0)

$$-\sigma_z = p \qquad (r < a)$$
$$= p/2 \qquad (r = a)$$
$$= 0 \qquad (r > a)$$

$$-\frac{\sigma_r+\sigma_\theta}{2} = -\frac{pa}{2}\int_0^\infty J_0(mr).J_1(ma)\left[A_1+B_1+(1+4\mu_1)C_1-(1+4\mu_1)D_1\right]dm$$

$$= -\frac{pa}{2}\int_0^\infty J_0(mr).J_1(ma)\left[-(1+2\mu_1)+4(1+\mu_1)A_1+8\mu_1(1+\mu_1)C_1\right]dm$$

$$= \frac{pa}{2}\int_0^\infty J_0(mr).J_1(ma)\,(1+2\mu_1)\,dm$$

$$-2pa(1+\mu_1)\int_0^\infty J_0(mr)J_1(ma)\left[(A_1e^x)+2\mu_1(C_1e^x)\right]e^{-x}dm$$

with

$$\frac{pa}{2}(1+2\mu_1)\int_0^\infty J_0(mr).J_1(ma)\,dm = \begin{cases}(1+2\mu_1)\frac{p}{2} & (r<a)\\ (1+2\mu_1)\frac{p}{4} & (r=a)\\ 0 & (r>a)\end{cases}$$

$$-\frac{\sigma_r-\sigma_\theta}{2} = -\frac{pa}{2}\int_0^\infty J_0(mr)\,J_1(ma)\left[A_1+B_1+C_1-D_1\right]dm$$

$$+pa\int_0^\infty \frac{J_1(mr).J_1(ma)}{mr}\left[A_1+B_1+C_1-D_1\right]dm$$

$$= \frac{pa}{2}(1-2\mu_1)\int_0^\infty J_0(mr)J_1(ma)\,dm$$

$$-pa(1-2\mu_1)\int_0^\infty \frac{J_1(mr).J_1(ma)}{mr}\,dm$$

$$-2pa(1-\mu_1)\int_0^\infty\left[J_0(mr)-\frac{2J_1(mr)}{mr}\right]J_1(ma)\left[(A_1e^x)+2\mu_1(C_1e^x)\right]e^{-x}dm$$

with

$$\frac{pa}{2}(1-2\mu_1)\int_0^\infty J_0(mr).J_1(ma)\,dm = \begin{cases} (1-2\mu_1)\frac{p}{2} & (r<a) \\ (1-2\mu_1)\frac{p}{4} & (r=a) \\ 0 & (r>a) \end{cases}$$

$$-pa(1-2\mu_1)\int_0^\infty \frac{J_1(mr).J_1(ma)}{mr}\,dm = \begin{cases} -(1-2\mu_1)\frac{p}{2} & (r<a) \\ -(1-2\mu_1)\frac{pa^2}{2r^2} & (r>a) \end{cases}$$

$$-\tau_{rz} = -pa\int_0^\infty J_1(mr).J_1(ma)\left[A_1 - B_1 + 2\mu_1 C_1 + 2\mu_1 D_1\right] dm$$

$$= 0$$

$$-w = +\frac{1+\mu_1}{E_1}pa\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}\left[A_1 - B_1 - (2-4\mu_1)C_1 - (2-4\mu_1)D_1\right] dm$$

$$= +\frac{2(1-\mu_1^2)}{E_1}.pa\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}\,dm$$

$$-\frac{4(1-\mu_1^2)}{E_1}pa\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}\left[(A_1e^x) - (1-2\mu_1)(C_1e^x)\right]e^{-x}\,dm$$

with

$$\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}\,dm = \begin{cases} 1 & (r=0) \\ F\left(\frac{1}{2},-\frac{1}{2};1;r^2/a^2\right) & (r<a) \\ 2/\pi & (r=a) \\ \frac{a}{2r}F\left(\frac{1}{2},\frac{1}{2};2;\frac{a^2}{r^2}\right) & (r>a) \end{cases}$$

$$-u = -\frac{1+\mu_1}{E_1}.pa.\int_0^\infty \frac{J_1(mr).J_0(ma)}{m}\left[A_1 + B_1 + C_1 - D_1\right] dm$$

$$u = \frac{1+\mu_1}{E_1}(1-2\mu_1)\cdot pa\int_0^\infty \frac{J_1(mr).J_1(ma)}{m}dm$$

$$-\frac{4(1-\mu_1^2)}{E_1}\cdot pa\int_0^\infty \frac{J_1(mr).J_1(ma)}{m}\left[(A_1e^x)+2\mu_1(C_1e^x)\right]e^{-x}dm.$$

with

$$\int_0^\infty \frac{J_1(mr).J_1(ma)}{m}dm = \begin{cases} \frac{r}{2a} & (r<a) \\ \frac{a}{2r} & (r>a) \end{cases}$$

4.3. Stresses and displacements in the first layer ($0<h<H_1$):

$$-\sigma_z = pa\int_0^\infty J_0(mr).J_1(ma)\left[A_1e^{mh}+B_1e^{-mh}-C_1(1-2\mu_1-mh)e^{mh}\right.$$
$$\left.+D_1(1-2\mu_1+mh)e^{-mh}\right]dm$$

$$= pa\int_0^\infty J_0(mr).J_1(ma)e^{-mh}dm + pa\int_0^\infty J_0(mr)J_1(ma)mhe^{-mh}$$
$$+pa\int_0^\infty J_0(mr).J_1(ma)\left[(A_1e^x).e^{-(x-mh)}-(C_1e^x)(1-2\mu_1-mh)e^{-(x-mh)}\right.$$
$$\left.-(A_1e^x)(1+2mh)e^{-(x+mh)}+(C_1e^x)(1-2\mu_1+mh-4\mu_1mh)e^{-(x+mh)}\right]dm$$

with

$$pa\int_0^\infty J_0(mr)J_1(ma)e^{-mh}dm = p\cdot\frac{\sqrt{a^2+h^2}-h}{\sqrt{a^2+h^2}} \qquad (r=0)$$

$$= \frac{p}{2\pi}\int_{-a}^{a}\frac{2h(a^2-x^2)^{1/2}}{(h^2+x^2+r^2-2xr)(h^2+r^2+a^2-2xr)^{1/2}}\cdot dx \qquad (r\neq 0)$$

$$pa\int_0^\infty J_0(mr)\,.\,J_1(ma)\,mh\,e^{-mh}\,dm = p\,.\,\frac{a^2 z}{(a^2+z^2)^{3/2}} \qquad (r=0)$$

$$= \frac{p}{2\pi}\int_{-a}^{a}\frac{2h\,(a^2-x^2)^{1/2}}{(h^2+x^2+r^2-2rx)\,(h^2+r^2+a^2-2rx)^{1/2}}\,dx$$

$$+ \frac{p}{2\pi}\int_{-a}^{a}\frac{2h\,(a^2-x^2)^{1/2}}{(h^2+r^2+a^2-2xr)^{3/2}}\,dx$$

$$- \frac{p}{2\pi}\int_{-a}^{a}\frac{2h\,(r^2+x^2-2rx)\,(a^2-x^2)^{1/2}}{(h^2+x^2+r^2-2xr)(h^2+r^2+a^2-2xr)^{1/2}}\left[\frac{1}{(r^2+a^2-2xr+h^2)}+\frac{2}{(h^2+x^2+r^2-2rx)}\right]dx$$

$$-\frac{\sigma_r+\sigma_\theta}{2} = -\frac{pa}{2}\int_0^\infty J_0(mr)\,.\,J_1(ma)\Big[A_1 e^{mh} + B_1 e^{-mh} + C_1(1+4\mu_1+mh)e^{mh}$$
$$- D_1(1+4\mu_1-mh)\,e^{-mh}\Big]\,dm$$

$$= pa\,.\,\frac{(1+2\mu)}{2}\int_0^\infty J_0(mr)\,J_1(ma)\,e^{-mh}\,dm - \frac{pa}{2}\int_0^\infty J_0(mr)\,J_1(ma)\,mh\,e^{-mh}\,dm$$

$$-\frac{pa}{2}\int_0^\infty J_0(mr)\,.\,J_1(ma)\Big[(A_1 e^{\chi})\,e^{-(\chi-mh)} + (C_1 e^{\chi})\,(1+4\mu_1+mh)\,e^{-(\chi-mh)}$$
$$+(A_1 e^{\chi})\,(3+4\mu_1-2mh)\,e^{-(\chi+mh)} - (C_1 e^{\chi})\,(1-4\mu_1-8\mu_1^2-mh+4\mu_1 mh)\,e^{-(\chi+mh)}\Big]\,dm$$

$$-\frac{\sigma_r-\sigma_\theta}{2} = -\frac{pa}{2}\int_0^\infty J_0(mr)\,J_1(ma)\Big[A_1 e^{mh} + B_1 e^{-mh} + C_1(1+mh)e^{mh}$$
$$- D_1(1-mh)\,e^{-mh}\Big]\,dm$$

$$+ pa\int_0^\infty \frac{J_1(mr)\,J_1(ma)}{mr}\Big[A_1 e^{mh} + B_1 e^{-mh} + C_1(1+mh)\,e^{mh}$$
$$- D_1(1-mh)\,e^{-mh}\Big]\,dm$$

$$\frac{\sigma_r - \sigma_\theta}{2} = pa\frac{(1-2\mu_1)}{2}\int_0^\infty J_1(mr).J_1(ma)e^{-mh}dm - \frac{pa}{2}\int_0^\infty J_0(mr).J_1(ma)mhe^{-mh}dm$$

$$-pa(1-2\mu_1)\int_0^\infty \frac{J_1(mr).J_1(ma)}{mr}e^{-mh}dm + pa\int_0^\infty \frac{J_1(mr).J_1(ma)}{mr}mhe^{-mh}dm$$

$$-pa\int_0^\infty \left[\frac{J_0(mr)}{2} - \frac{J_1(mr)}{mr}\right].\ J_1(ma)\Big[(A_1e^x)\,e^{-(x-mh)}$$

$$+ (C_1e^x)(1+mh)e^{-(x-mh)} + (A_1e^x)(3-4\mu_1-2mh)\,e^{-(x+mh)}$$

$$-(C_1e^x)(1-8\mu_1+8\mu_1^2-mh+4\mu_1mh)\,e^{-(x+mh)}\Big]\,dm$$

with

$$pa\int_0^\infty \frac{J_1(mr).J_1(ma)}{mr}e^{-mh}dm = \frac{p}{2}\cdot\frac{\sqrt{a^2+h^2}-h}{\sqrt{a^2+h^2}} \qquad (r=0)$$

$$= p.\frac{a^2}{\pi}\int_0^\pi \frac{1}{\omega^2}\left[1-\frac{h}{(h^2+\omega^2)^{1/2}}\right]\sin^2\phi\, d\phi \qquad (r\neq 0)$$

$$\omega^2 = a^2+r^2-2ar\cos\phi$$

$$pa\int_0^\infty \frac{J_1(mr).J_1(ma)}{mr}mhe^{-mh}dm = \frac{p}{2}\,\frac{a^2h}{(a^2+h^2)^{3/2}} \qquad (r=0)$$

$$= p\frac{a^2}{\pi}\int_0^\pi \frac{h}{(h^2+\omega^2)^{3/2}}\sin^2\phi\, d\phi \qquad (r\neq 0)$$

$$-\tau_{rz} = -pa\int_0^\infty J_1(mr).J_1(ma)\Big[A_1e^{mh} - B_1e^{-mh} + C_1(2\mu_1+mh)e^{mh}$$

$$+ D_1(2\mu_1-mh)\,e^{-mh}\Big]\,dm.$$

$$\tau_{rz} = pa\int_0^\infty J_1(mr)\,J_1(ma)\,mh\,e^{-mh}\,dm$$

$$-pa\int_0^\infty J_1(mr).J_1(ma)\Big[(A_1e^x)\,e^{-(x-mh)} + (C_1e^x)(2\mu_1+mh)\,e^{-(x-mh)}$$

$$- (A_1e^x)(1-2mh)\,e^{-(x+mh)} - (C_1e^x)(2\mu_1+mh-4\mu_1mh)\,e^{-(x+mh)}\Big]dm$$

with

$$pa\int_0^\infty J_1(mr).J_1(ma)\,mh\,e^{-mh}\,dm = 0 \qquad (r=0)$$

$$= \frac{3pa^2r}{\pi}\int_0^\pi \frac{h^2}{(h^2+\omega^2)^{5/2}}\cdot \sin^2\phi\,d\phi \qquad (r\neq 0)$$

$$-w = +\frac{1+\mu_1}{E_1}pa\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}\Big[A_1e^{mh} - B_1e^{-mh} - C_1(2-4\mu_1-mh)\,e^{mh}$$

$$- D_1(2-4\mu_1+mh)\,e^{-mh}\Big]dm$$

$$= +\frac{2(1-\mu_1^2)}{E_1}pa\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}e^{-mh}\,dm - \frac{(1+\mu_1)}{E_1}pa\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}mh\,e^{-mh}\,dm$$

$$- \frac{(1+\mu_1)}{E_1}pa\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}\Big[(A_1e^x)\,e^{-(x-mh)} - (C_1e^x)(2-4\mu_1-mh)\,e^{-(x-mh)}$$

$$+(A_1e^x)(3-4\mu_1+2mh)\,e^{-(x+mh)} - (C_1e^x)(2-8\mu_1+8\mu_1^2+mh-4\mu_1mh)\,e^{-(x+mh)}\Big]dm$$

with

$$pa\int_0^\infty \frac{J_0(mr).J_1(ma)}{m}e^{-mh}\,dm = p\left[(a^2+h^2)^{1/2} - h\right] \qquad (r=0)$$

$$= \frac{p}{2\pi}\int_{-a}^{a} \ln\frac{(r^2-2xr+a^2+h^2)^{1/2} + (a^2-x^2)^{1/2}}{(r^2-2xr+a^2+h^2)^{1/2} - (a^2-x^2)^{1/2}}\,dx \qquad (r\neq 0)$$

$$-u = -\frac{1+\mu_1}{E_1}pa\int_0^\infty \frac{J_1(mr).J_1(ma)}{m}\Big[A_1e^{mh} + B_1e^{-mh} + C_1(1+mh)e^{mh} - D_1(1-mh)e^{-mh}\Big]dm$$

$$u = \frac{(1+\mu_1)(1-2\mu_1)}{E_1} pa \int_0^\infty \frac{J_1(mr).J_1(ma)}{m} e^{-mh} dm$$

$$- \frac{(1+\mu_1)}{E_1} pa \int_0^\infty \frac{J_1(mr).J_1(ma)}{m} . mh e^{-mh} dm$$

$$- \frac{(1+\mu_1)}{E_1} pa \int_0^\infty \frac{J_1(mr).J_1(ma)}{m} \Big[(A_1 e^x) e^{-(x-mh)} + (C_1 e^x)(1+mh) e^{-(x-mh)}$$

$$+ (A_1 e^x)(3-4\mu_1-2mh) e^{-(x+mh)} - (C_1 e^x)(1-8\mu_1+8\mu_1^2-mh+4\mu_1 mh) e^{-(x+mh)} \Big] dm$$

4.4. Stresses and displacements in the other layers.

The relations for the stresses and displacements are given in paragraph 4.1. The terms $A_i m^2 e^{mz}$, $B_i m^2 e^{-mz}$, $C_i m e^{mz}$ and $D_i m e^{-mz}$ are to be replaced

In the second layer ($H_1 < h < H_1 + H_2$) by

$$(A_2 e^y) . e^{-(y-mh)} \quad (B_2 e^{-x}) e^{-(mh-x)} \quad (C_2 e^y) e^{-(y-mh)} \quad (D_2 e^{-x}) e^{-(mh-x)}$$

In the third layer ($H_1 + H_2 < h < H_1 + H_2 + H_3$) by

$$(A_3 e^z) . e^{-(z-mh)} \quad (B_3 e^{-y}) e^{-(mh-y)} \quad (C_3 e^z) e^{-(z-mh)} \quad (D_3 e^{-y}) e^{-(mh-y)}$$

In the fourth layer ($H_1 + H_2 + H_3 < h < H_1 + H_2 + H_3 + H_4$) by

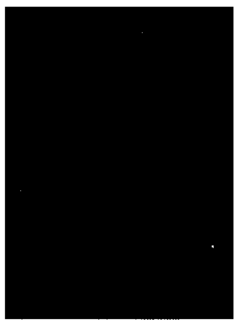
$$B_4 e^{-mh} \longrightarrow (B_4 e^{-z}) . e^{-(mh-z)}$$

$$D_4 e^{-mh} \longrightarrow (D_4 e^{-z}) . e^{-(mh-z)}$$

$$A_4 e^{mh} = (A_4 e^t) e^{-(t-mh)}$$

$$= (B_4 e^{-z}) e^{-(t-z)} . e^{-(t-mh)}$$

$$+ (2-4\mu_4+t)(D_4 e^{-z}) . e^{-(t-z)} . e^{-(t-mh)}$$



APPENDIX 2

ALGEBRAICAL ANALYSIS OF ISOTROPIC LAYERED SYSTEMS WITH FIXED BOTTOM AND FULL SLIP CONDITIONS AT THE TWO FIRST INTERFACES

SYMBOLS

We write

$A_i = A_i m^2 \qquad B_i = B_i m^2 \qquad C_i = C_i m \qquad D_i = D_i m$

$$F_1 = \frac{E_1(1+\mu_2)}{E_2(1+\mu_1)}$$

$$K_1 = \frac{E_2(1+\mu_3)}{E_3(1+\mu_2)}$$

$$L_1 = \frac{E_3(1+\mu_4)}{E_4(1+\mu_2)}$$

$$x = m H_1$$

$$y = m(H_1 + H_2)$$

$$z = m(H_1 + H_2 + H_3)$$

$$t = m(H_1 + H_2 + H_3 + H_4)$$

where H_1, H_2, H_3, H_4 are the thicknesses of the successive layers. The index 1 applies to the surface layer.

We shall successively analyse a two layered, a three layered and a four layered structure.

Chapter 1. The two layered structure.

1. The boundary conditions.

Boundary conditions at the surface (z = 0):

$$\sigma_z: \quad A_1 + B_1 - C_1(1-2\mu_1) + D_1(1-2\mu_1) = 1$$

$$\tau_{rz}: \quad A_1 - B_1 + C_1 2\mu_1 + D_1 2\mu_1 = 0$$

Boundary conditions at the interface ($z = H_1$):

$$\sigma_z: \quad A_1 e^{x} + B_1 e^{-x} - C_1(1-2\mu_1-x)e^{x} + D_1(1-2\mu_1+x)e^{-x} =$$
$$A_2 e^{x} + B_2 e^{-x} - C_2(1-2\mu_2-x)e^{x} + D_2(1-2\mu_2+x)e^{-x}$$

$$\tau_{rz}: \quad A_1 e^{x} - B_1 e^{-x} + C_1(2\mu_1+x)e^{x} + D_1(2\mu_1-x)e^{-x} = 0$$

$$A_2 e^{x} - B_2 e^{-x} + C_2(2\mu_2+x)e^{x} + D_2(2\mu_2-x)e^{-x} = 0$$

$$w: \quad A_1 e^{x} - B_1 e^{-x} - C_1(2-4\mu_1-x)e^{x} - D_1(2-4\mu_1+x)e^{-x} =$$
$$F_1\left[A_2 e^{x} - B_2 e^{-x} - C_2(2-4\mu_2-x)e^{x} - D_2(2-4\mu_2+x)e^{-x}\right]$$

Boundary conditions at the bottom ($z = H_1 + H_2$):

$$w: \quad A_2 e^{y} - B_2 e^{-y} - C_2(2-4\mu_2-y)e^{y} - D_2(2-4\mu_2+y)e^{-y} = 0$$

$$C_2 = 0$$

2. Resolution of the system of 6 boundary equations.

Adding and substracting the surface conditions we obtain

$$2A_1 = 1 + C_1(1-4\mu_1) - D_1$$

$$2B_1 = 1 + C_1 - D_1(1-4\mu_1)$$

Adding and substracting the first two conditions at the interface, we obtain

$$2A_1 e^{x} - C_1(1-4\mu_1-2x)e^{x} + D_1 e^{-x} = [A2]$$

$$2B_1 e^{-x} - C_1 e^{x} + D_1(1-4\mu_1+2x)e^{-x} = [A2]$$

where

$$[A2] = A_2 e^{x} + B_2 e^{-x} - C_2(1-2\mu_2 - x)e^{x} + D_2(1-2\mu_2+x)e^{-x}$$

and, taken in account the bottom conditions,

$$[A2] = \left(1+e^{-2(y-x)}\right) B_2 e^{-x} + \left[(1-2\mu_2+x) + (2-4\mu_2+y)e^{-2(y-x)}\right] D_2 e^{-x}$$

We replace A_1 and B_1 by their values in function of C_1 and D_1 and solve the system

$$C_1 e^{x} = \frac{[A2]\left[(2x-1)e^{-2x}+1\right] - (1+2x)e^{-x} + e^{-3x}}{2(1+2x^2)e^{-2x} - e^{-4x} - 1}$$

$$D_1 e^{-x} = \frac{[A2]\left[(1+2x)e^{-2x} - e^{-4x}\right] + (1-2x)e^{-3x} - e^{-x}}{2(1+2x^2)e^{-2x} - e^{-4x} - 1}$$

We combine the τ_{rz}- condition with the w-condition so that

$$2(1-\mu_1)C_1 e^{x} + 2(1-\mu_1)D_1 e^{-x} = F_1 \cdot 2(1-\mu_2) D_2 e^{-x}$$

and with

$$F_2 = F_1 \frac{(1-\mu_2)}{(1-\mu_1)} \qquad C_1 e^{x} + D_1 e^{-x} = F_2 \cdot D_2 e^{-x}$$

We replace C_1 and D_1 by their values in function of B_2 and D_2

$$\frac{[A2]\left[1+4xe^{-2x} - e^{-4x}\right] - 2(1+x)e^{-x} + 2(1-x)e^{-3x}}{2(1+2x^2)e^{-2x} - e^{-4x} - 1} = F_2 \cdot D_2 e^{-x}$$

We take the τ_{rz} -condition and solve the system

$$B_2 e^{-x}\left[1+e^{-2(y-x)}\right] + D_2 e^{-x}\left[(1-2\mu_2+x-R_3) + (2-4\mu_2+y)e^{-2(y-x)}\right] = -\frac{R_2}{R_1}$$

$$B_2 e^{-x}\left[e^{-2(y-x)} - 1\right] + D_2 e^{-x}\left[(2\mu_2 - x) + (2-4\mu_2+y)e^{-2(y-x)}\right] = 0$$

where

$$R_1 = 1 + 4x e^{-2x} - e^{-4x}$$

$$R_2 = 2(1-x) e^{-3x} - 2(1+x) e^{-x}$$

$$\nabla_1 = 2(1+2x^2) e^{-2x} - e^{-4x} - 1$$

$$R_3 = F_2 \cdot \frac{\nabla_1}{R_1}$$

$$B_2 e^{-y} = -\frac{R_2}{R_1} \; \frac{(2-4\mu_2+y) e^{-2(y-x)} + (2\mu_2 - x)}{\nabla_2}$$

$$D_2 e^{-y} = \frac{R_2}{R_1} \; \frac{e^{-2(y-x)} - 1}{\nabla_2}$$

$$\nabla_2 = (3 - 4\mu_2 + 2y - 2x + R_3) e^{-2(y-x)} + 1 - R_3$$

3. Relations for the parameters.

$$C_1 e^{x} = \frac{[A2]\left[(2x-1) e^{-2x} + 1\right] - (1+2x) e^{-x} + e^{-3x}}{\nabla_1}$$

$$D_1 e^{-x} = \frac{[A2]\left[(1+2x) e^{-2x} - e^{-4x}\right] + (1-2x) e^{-3x} - e^{-y}}{\nabla_1}$$

$$A_1 e^{x} = \frac{1}{2}\left\{ [A2] + (1-4\mu_1 - 2x) C_1 e^{x} - D_1 e^{-x} \right\}$$

$$B_1 = 2\mu_1 + (1-4\mu_1)(A_1 e^{x}) e^{-x} + 4\mu_1 (1-2\mu_1)(C_1 e^{x}) e^{-x}$$

$$D_1 = 1 - 2(A_1 e^{x}) e^{-x} + (1-4\mu_1)(C_1 e^{x}) e^{-x}$$

Chapter 2. The three layered structure.

1. Supplementary boundary conditions.

Boundary conditions at the second interface ($z = H_1 + H_2$):

$$\sigma_z: \quad A_2 e^{y} + B_2 e^{-y} - C_2 (1-2\mu_2-y) e^{y} + D_2 (1-2\mu_2+y) e^{-y} =$$
$$A_3 e^{y} + B_3 e^{-y} - C_3 (1-2\mu_3-y) e^{y} + D_3 (1-2\mu_3+y) e^{-y}$$

$$\tau_{rz}: \quad A_2 e^{y} - B_2 e^{-y} + C_2 (2\mu_2+y) e^{y} + D_2 (2\mu_2-y) e^{-y} = 0$$
$$A_3 e^{y} - B_3 e^{-y} + C_3 (2\mu_3+y) e^{y} + D_3 (2\mu_3-y) e^{-y} = 0$$

$$w: \quad A_2 e^{y} - B_2 e^{-y} - C_2 (2-4\mu_2-y) e^{y} - D_2 (2-4\mu_2+y) e^{-y} =$$
$$K_1 \left[A_3 e^{y} - B_3 e^{-y} - C_3 (2-4\mu_3-y) e^{y} - D_3 (2-4\mu_3+y) e^{-y} \right]$$

Boundary conditions at the bottom ($z = H_1 + H_2 + H_3$):

$$w: \quad A_3 e^{z} - B_3 e^{-z} - C_3 (2-4\mu_3-z) e^{z} - D_3 (2-4\mu_3+z) e^{-z} = 0$$

$$C_3 = 0$$

2. Resolution of the system of 10 boundary equations.

Adding and substracting the surface conditions, we obtain

$$2A_1 = 1 + C_1 (1-4\mu_1) - D_1$$
$$2B_1 = 1 + C_1 - D_1 (1-4\mu_1)$$

We add the first two conditions at the first interface

$$2A_1 e^{x} - C_1 (1-4\mu_1-2x) e^{x} + D_1 e^{-x} = [A2]$$
$$2B_1 e^{-x} - C_1 e^{x} + D_1 (1-4\mu_1+2x) e^{-x} = [A2]$$

where

$$[A2] = A_2 e^{x} + B_2 e^{-x} - C_2 (1-2\mu_2-x) e^{x} + D_2 (1-2\mu_2+x) e^{-x}$$

We replace A_1 and B_1 by their values in function of C_1 and D_1 and solve the system

$$C_1 e^{x} = \frac{[A2]\left[(2x-1)e^{-2x}+1\right] - (1+2x)e^{-x} + e^{-3x}}{2(1+2x^2)e^{-2x} - e^{-4x} - 1}$$

$$D_1 e^{-x} = \frac{[A2]\left[(1+2x)e^{-2x} - e^{-4x}\right] + (1-2x)e^{-3x} - e^{-x}}{2(1+2x^2)e^{-2x} - e^{-4x} - 1}$$

We combine the τ_{rz}-conditions with the w-condition so that

$$2(1-\mu_1)C_1 e^{x} + 2(1-\mu_1)D_1 e^{-x} = F_1\left[2(1-\mu_2)C_2 e^{x} + 2(1-\mu_2)D_2 e^{-x}\right]$$

and with

$$F_2 = F_1 \frac{(1-\mu_2)}{(1-\mu_1)}$$

$$C_1 e^{x} + D_1 e^{-x} = F_2\left[C_2 e^{x} + D_2 e^{-x}\right]$$

We replace C_1 and D_1 by their values in function of A_2, B_2, C_2 and D_2

$$\frac{[A2]\left[1+4xe^{-2x}-e^{-4x}\right] - 2(1+x)e^{-x} + 2(1-x)e^{-3x}}{2(1+2x^2)e^{-2x} - e^{-4x} - 1} = F_2\left[C_2 e^{x} + D_2 e^{-x}\right]$$

Together with the τ_{rz}-condition, we obtain then the system

$$A_2 e^{x} - B_2 e^{-x} + C_2(2\mu_2 + x)e^{x} + D_2(2\mu_2 - x)e^{-x} = 0$$

$$A_2 e^{x} + B_2 e^{-x} - C_2(1-2\mu_2 - x + R_3)e^{x} + D_2(1-2\mu_2 + x - R_3)e^{-x} = -\frac{R_2}{R_1}$$

where

$$R_1 = 1 + 4xe^{-2x} - e^{-4x}$$

$$R_2 = 2(1-x)e^{-3x} - 2(1+x)e^{-x}$$

$$V_1 = 2(1+2x^2)e^{-2x} - e^{-4x} - 1$$

$$R_3 = \frac{F_2 \cdot V_1}{R_1}$$

We solve the system for A_2 and B_2

$$2A_2 = -\frac{R_2}{R_1}e^{x} + C_2(1-4\mu_2-2x+R_3) - D_2(1-R_3)e^{-2x}$$

$$2B_2 = -\frac{R_2}{R_1}e^{x} + C_2(1+R_3)e^{2x} - D_2(1-4\mu_2+2x-R_3)$$

We add and substract the first two conditions at the second interface

$$2A_2e^{y} - C_2(1-4\mu_2-2y)e^{y} + D_2e^{-y} = [A3]$$

$$2B_2e^{-y} - C_2e^{y} + D_2(1-4\mu_2+2y) = [A3]$$

where

$$[A3] = A_3e^{y} - B_3e^{-y} - C_3(1-2\mu_3-y)e^{y} + D_3(1-2\mu_3+y)e^{-y}$$

and, taken in account the bottom conditions,

$$[A3] = \left[1 + e^{-2(z-y)}\right]B_3 \cdot e^{-y} + \left[(1-2\mu_3+y) + (2-4\mu_3+2)e^{-2(z-y)}\right]D_3e^{-y}$$

We replace A_2 and B_2 by their values in function of C_2 and D_2 and solve the system

$$C_2e^{y} = \frac{[A3]\left[(2y-2x+R_3-1)e^{-2(y-x)} + (1-R_3)\right] + \frac{R_2}{R_1}\left[(1+2y-2x)e^{-(y-x)} - e^{-3(y-x)}\right]}{2\left[1+2(y-x)^2\right]e^{-2(y-x)} + 4R_3(y-x)e^{-2(y-x)} - (1+R_3)e^{-4(y-x)} - (1-R_3)}$$

$$D_2e^{-y} = \frac{[A3]\left[(1+2y-2x+R_3)e^{-2(y-x)} - (1+R_3)e^{-4(y-x)}\right] + \frac{R_2}{R_1}\left[e^{-(y-x)} - (1-2y+2x)e^{-3(y-x)}\right]}{2\left[1+2(y-x)^2\right]e^{-2(y-x)} + 4R_3(y-x)e^{-2(y-x)} - (1+R_3)e^{-4(y-x)} - (1-R_3)}$$

We combine the τ_{rz}-conditions with the w-conditions so that

$$2(1-\mu_2)C_2e^{y} + 2(1-\mu_2)D_2e^{-y} = k_1\left[2(1-\mu_3)D_3e^{-y}\right]$$

and with

$$k_2 = k_1\frac{(1-\mu_3)}{(1-\mu_2)}$$

$$C_2e^{y} + D_2e^{-y} = k_2D_3e^{-y}$$

We replace C_2 and D_2 by their values in function of B_3 and D_3

$$\frac{[A3]\left[2(2y-2x+R_3)e^{-2(y-x)}+(1-R_3)-(1+R_3)e^{-4(y-x)}\right]+\frac{2R_2}{R_1}\left[(1+y-x)e^{-(y-x)}-(1-y+x)e^{-3(y-x)}\right]}{2\left[1+2(y-x)^2\right]e^{-2(y-x)}+4R_3(y-x)e^{-2(y-x)}-(1+R_3)e^{-4(y-x)}-(1-R_3)}$$

$$= K_2 D_3 e^{-y}$$

Together with the τ_{rz}-condition, we obtain then the system

$$B_3e^{-y}\left[e^{-2(z-y)}-1\right]+D_3e^{-y}\left[(2-4\mu_3+z)e^{-2(z-y)}+(2\mu_3-y)\right]=0$$

$$B_3e^{-y}\left[e^{-2(z-y)}+1\right]+D_3e^{-y}\left[(2-4\mu_3+z)e^{-2(z-y)}+(1-2\mu_3+y)-\vartheta_3\right]=-\frac{\vartheta_2}{\vartheta_1}$$

where

$$\vartheta_1 = 2(2y-2x+R_3)e^{-2(y-x)}+(1-R_3)-(1+R_3)e^{-4(y-x)}$$

$$\vartheta_2 = \frac{2R_2}{R_1}\left[(1+y-x)e^{-(y-x)}-(1-y+x)e^{-3(y-x)}\right]$$

$$\nabla_2 = 2\left[1+2(y-x)^2\right]e^{-2(y-x)}+4R_3(y-x)e^{-2(y-x)}-(1+R_3)e^{-4(y-x)}-(1-R_3)$$

$$\vartheta_3 = \frac{K_2 . \nabla_2}{\vartheta_1}$$

We solve the system

$$B_3e^{-y} = -\frac{\vartheta_2}{\vartheta_1}\cdot\frac{(2-4\mu_3+z)e^{-2(z-y)}+(2\mu_3-y)}{\nabla_3}$$

$$D_3e^{-y} = \frac{\vartheta_2}{\vartheta_1}\cdot\frac{e^{-2(z-y)}-1}{\nabla_3}$$

$$\nabla_3 = \left[3-4\mu_3+2z-2y+\vartheta_3\right]e^{-2(z-y)}+1-\vartheta_3$$

3. Relations for the parameters.

We have

$$C_2 e^{y} = \frac{[A3]\left[(2y-2x+R_3-1)e^{-2(y-x)} + (1-R_3)\right] + \frac{R_2}{R_1}\left[(1+2y-2x)e^{-(y-x)} - e^{-3(y-x)}\right]}{\nabla_2}$$

$$D_2 e^{-y} = \frac{[A3]\left[(1+2y-2x+R_3)e^{-2(y-x)} - (1+R_3)e^{-4(y-x)}\right] + \frac{R_2}{R_1}\left[e^{-(y-x)} - (1-2y+2x)e^{-3(y-x)}\right]}{\nabla_2}$$

so that

$$D_2 e^{-x} = D_2 e^{-y} . e^{(y-x)}$$

$$= \frac{[A3]\left[(1+2y-2x+R_3)e^{-(y-x)} - (1+R_3)e^{-3(y-x)}\right] + \frac{R_2}{R_1}\left[1 - (1-2y+2x)e^{-2(y-x)}\right]}{\nabla_2}$$

We have

$$2A_2 e^{y} - C_2(1-4\mu_2-2y)e^{y} + D_2 e^{-y} = [A3]$$

so that

$$A_2 e^{y} = \frac{1}{2}\left\{[A3] + C_2(1-4\mu_2-2y)e^{y} - D_2 e^{-x} . e^{-(y-x)}\right\}$$

We have

$$2B_2 = -\frac{R_2}{R_1}e^{x} + C_2(1+R_3)e^{2x} - D_2(1-4\mu_2+2x-R_3)$$

so that

$$B_2 e^{-x} = \frac{1}{2}\left[-\frac{R_2}{R_1} + C_2 e^{y}(1+R_3)e^{-(y-x)} - D_2 e^{-x}(1-4\mu_2+2x-R_3)\right]$$

We have

$$C_1 e^{x} = \frac{[A2].\left[(2x-1)e^{-2x}+1\right] - (1+2x)e^{-x} + e^{-3x}}{\nabla_1}$$

$$D_1 e^{-x} = \frac{[A2]\left[(1+2x)e^{-2x} - e^{-4x}\right] + (1-2x)e^{-3x} - e^{-x}}{\nabla_1}$$

$$2A_1 e^{x} - C_1(1-4\mu_1-2x)e^{x} + D_1 e^{-x} = [A2]$$

so that

$$A_1 e^{x} = \frac{1}{2}\left\{[A2] + C_1 e^{x}(1-4\mu_1-2x) - D_1 e^{-x}\right\}$$

Using the surface conditions, we have finally

$$B_1 = 2\mu_1 + A_1 e^{x}(1-4\mu_1)e^{-x} + C_1 e^{x}.4\mu_1(1-2\mu_1)e^{-x}$$

$$D_1 = 1 - 2A_1 e^{x}.e^{-x} + C_1 e^{x}(1-4\mu_1)e^{-x}.$$

Chapter 3. The four layered structure.

1. Supplementary boundary conditions.

Boundary conditions at the third interface ($z = H_1 + H_2 + H_3$):

$$\sigma_z: \quad A_3 e^{z} + B_3 e^{-z} - C_3 (1-2\mu_3 - z) e^{z} + D_3 (1-2\mu_3 + z) e^{-z} =$$

$$A_4 e^{z} + B_4 e^{-z} - C_4 (1-2\mu_4 - z) e^{z} + D_4 (1-2\mu_4 + z) e^{-z}$$

$$\tau_{rz}: \quad A_3 e^{z} - B_3 e^{-z} + C_3 (2\mu_3 + z) e^{z} + D_3 (2\mu_3 - z) e^{-z} =$$

$$A_4 e^{z} - B_4 e^{-z} + C_4 (2\mu_4 + z) e^{z} + D_4 (2\mu_4 - z) e^{-z}$$

$$w: \quad A_3 e^{z} - B_3 e^{-z} - C_3 (2-4\mu_3 - z) e^{z} - D_3 (2-4\mu_3 + z) e^{-z} =$$

$$L \left[A_4 e^{z} - B_4 e^{-z} - C_4 (2-4\mu_4 - z) e^{z} - D_4 (2-4\mu_4 + z) e^{-z} \right]$$

$$u: \quad A_3 e^{z} + B_3 e^{-z} + C_3 (1+z) e^{z} - D_3 (1-z) e^{-z} =$$

$$L \left[A_4 e^{z} + B_4 e^{-z} + C_4 (1+z) e^{z} - D_4 (1-z) e^{-z} \right]$$

Boundary conditions at the bottom ($z = H_1 + H_2 + H_3 + H_4$):

$$w = 0$$

$$A_4 e^{t} - B_4 e^{-t} - C_4 (2-4\mu_4 - t) e^{t} - D_4 (2-4\mu_4 + t) e^{-t} = 0$$

$$C_4 = 0$$

2. Resolution of the system of 16 equations.

In the equations of the conditions at the third interface, A_4 is replaced by its value taken from the fixed bottom condition:

$$A_4 e^{h} = B_4 e^{-h} + D_4 (2-4\mu_4+h) e^{-h}$$

We write the conditions at the third interface in matrixform

$$M_5 (A_3 B_3 C_3 D_3)^T = M_6 (B_4 D_4)^T$$

We invert M_5

$$(A_3 B_3 C_3 D_3)^T = M_5^{-1} . M_6 (B_4 D_4)^T$$

where

$$M_5^{-1} = -\frac{1}{4(1-\mu_3)} . e^{-z} \begin{pmatrix} -(1+z) & -(2-4\mu_3-z) & -(2\mu_3+z) & -(1-2\mu_3-z) \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$-\frac{1}{4(1-\mu_3)} . e^{z} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -(1-z) & (2-4\mu_3+z) & (2\mu_3-z) & -(1-2\mu_3+z) \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 1 \end{pmatrix}$$

$$M_6 = e^{-h} \begin{pmatrix} 1 & (2-4\mu_4+h) \\ 1 & (2-4\mu_4+h) \\ L & L(2-4\mu_4+h) \\ L & L(2-4\mu_4+h) \end{pmatrix} + e^{-z} \begin{pmatrix} 1 & (1-2\mu_4+z) \\ -1 & (2\mu_4-z) \\ -L & -L(2-4\mu_4+z) \\ L & -L(1-z) \end{pmatrix}$$

$$(A_3 B_3 C_3 D_3)^T = -\frac{1}{4(1-\mu_3)} \left[e^{-z} . M_{51} + e^{z} . M_{52} \right] . \left[e^{-h} . M_{61} + e^{-z} . M_{62} \right] (B_4 D_4)^T$$

$$= -\frac{1}{4(1-\mu_3)} \Big[e^{-2h} . M_{5161} + e^{-2z} . M_{5162} + e^{2(h-z)} . M_{5261}$$

$$+ M_{5262} \Big] . (B_4 \quad D_4)^T$$

$$M_{51} . M_{61} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = M_{5161}$$

$$M_{51} . M_{62} = \begin{pmatrix} + & + \\ 0 & 0 \\ + & + \\ 0 & 0 \end{pmatrix} = M_{5162}$$

$$M_{52} . M_{61} = \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & + \end{pmatrix} = M_{5261}$$

$$M_{52} . M_{62} = \begin{pmatrix} 0 & 0 \\ L_1 & L_3 \\ 0 & 0 \\ 0 & L_2 \end{pmatrix} + 2(L_1 - L_2) \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} = M_{5262}$$

$$L_1 = -2(1-2\mu_3) - (1+L)$$
$$L_2 = -2L(1-2\mu_4) - (1+L)$$
$$L_3 = \frac{1}{2}\left[L_1(1-4\mu_4) - L_2(1-4\mu_3)\right]$$

We consider now the boundary conditions at the surface.
Adding and substracting the surface conditions, we obtain

$$2A_1 = 1 + C_1(1-4\mu_1) - D_1$$
$$2B_1 = 1 + C_1 - D_1(1-4\mu_1)$$

Adding and substracting then the first two conditions at the first interface, we obtain

$$2A_1e^{x} - C_1(1-4\mu_1-2x)e^{x} + D_1e^{-x} = [A2]$$
$$2B_1e^{-x} - C_1e^{x} + D_1(1-4\mu_1+2x)e^{-x} = [A2]$$

where

$$[A2] = A_2e^{x} + B_2e^{-x} - C_2(1-2\mu_2-x)e^{x} + D_2(1-2\mu_2+x)e^{-x}$$

We replace A_1 and B_1 by their values in function of C_1 and D_1 and solve the system

$$C_1e^{x} = \frac{[A2]\left[(2x-1)e^{-2x}+1\right] - (1+2x)e^{-x} + e^{-3x}}{2(1+2x^2)e^{-2x} - e^{-4x} - 1}$$

$$D_1e^{-x} = \frac{[A2]\left[(1+2x)e^{-2x} - e^{-4x}\right] + (1-2x)e^{-3x} - e^{-x}}{2(1+2x^2)e^{-2x} - e^{-4x} - 1}$$

We combine the τ_{rz}-conditions with the w-condition so that

$$2(1-\mu_1)C_1e^{x} + 2(1-\mu_1)D_1e^{-x} = F\left[2(1-\mu_2)C_2e^{x} + 2(1-\mu_2)D_2e^{-x}\right]$$

and with $F_1 = F\dfrac{(1-\mu_2)}{(1-\mu_1)}$

$$C_1e^{x} + D_1e^{-x} = F_1\left[C_2e^{x} + D_2e^{-x}\right]$$

We replace C_1 and D_1 by their values in function of A_2, B_2, C_2 and D_2

$$\frac{[A2]\left[1+4xe^{-2x}-e^{-4x}\right] - 2(1+x)e^{-x} + 2(1-x)e^{-3x}}{2(1+2x^2)e^{-2x} - e^{-4x} - 1} = F_1\left[C_2e^{x} + D_2e^{-x}\right]$$

Together with the τ_{rz}- condition, we obtain then the system

$$A_2e^{x} - B_2e^{-x} + C_2(2\mu_2+x)e^{x} + D_2(2\mu_2-x)e^{-x} = 0$$

$$A_2e^{x} + B_2e^{-x} - C_2(1-2\mu_2-x+R_3)e^{x} + D_2(1-2\mu_2+x-R_3)e^{-x} = -\frac{R_2}{R_1}$$

where

$$R_1 = 1 + 4xe^{-2x} - e^{-4x}$$

$$R_2 = 2(1-x)e^{-3x} - 2(1+x)e^{-x}$$

$$\nabla_1 = 2(1+2x^2)e^{-2x} - e^{-4x} - 1$$

$$R_3 = \frac{F_1 \cdot \nabla_1}{R_1}$$

We notice that for $m = \infty$

$$R_1 = 1 \qquad R_2 = 0 \qquad \nabla_1 = -1 \qquad R_3 = -F_1$$

We solve the system for A_2 and B_2

$$2A_2 = -\frac{R_2}{R_1}e^{-x} + C_2(1-4\mu_2-2x+R_3) - D_2(1-R_3)e^{-2x}$$

$$2B_2 = -\frac{R_2}{R_1}e^{x} + C_2(1+R_3)e^{2x} - D_2(1-4\mu_2+2x-R_3)$$

Adding and substracting now the first two conditions at the second interface, we obtain

$$2A_2e^{y} - C_2(1-4\mu_2-2y)e^{y} + D_2e^{-y} = [A3]$$

$$2B_2e^{-y} - C_2e^{y} + D_2(1-4\mu_2+2y)e^{-y} = [A3]$$

where

$$[A3] = A_3e^{y} + B_3e^{-y} - C_3(1-2\mu_3-y)e^{y} + D_3(1-2\mu_3+y)e^{-y}$$

We replace A_2 and B_2 by their values in function of C_2 and D_2 and solve the system.

$$C_2 e^{y} = \frac{[A_3]\left[(2y-2x+R_3-1)e^{-2(y-x)} + (1-R_3)\right] + \frac{R_2}{R_1}\left[(1+2y-2x)e^{-(y-x)} - e^{-3(y-x)}\right]}{2\left[1+2(y-x)^2\right]e^{-2(y-x)} + 4R_3(y-x)e^{-2(y-x)} - (1+R_3)e^{-4(y-x)} - (1-R_3)}.$$

$$D_2 e^{-y} = \frac{[A_3]\left[(1+2y-2x+R_3)e^{-2(y-x)} - (1+R_3)e^{-4(y-x)}\right] + \frac{R_2}{R_1}\left[e^{-(y-x)} - (1-2y+2x)e^{-3(y-x)}\right]}{2\left[1+2(y-x)^2\right]e^{-2(y-x)} + 4R_3(y-x)e^{-2(y-x)} - (1+R_3)e^{-4(y-x)} - (1-R_3)}$$

We combine the τ_{rz}- conditions with the w-condition so that

$$2(1-\mu_2)C_2e^{y} + 2(1-\mu_2)D_2e^{-y} = k\left[2(1-\mu_3)C_3e^{y} + 2(1-\mu_3)D_3e^{-y}\right]$$

and with

$$K_1 = k\frac{(1-\mu_3)}{(1-\mu_2)}$$

$$C_2e^{y} + D_2e^{-y} = K_1\left[C_3e^{y} + D_3e^{-y}\right]$$

We replace C_2 and D_2 by their values in function of A_3, B_3, C_3 and D_3

$$\frac{[A_3]\left[2(2y-2x+R_3)e^{-2(y-x)} + (1-R_3) - (1+R_3)e^{-4(y-x)}\right] + \frac{2R_2}{R_1}\left[(1+y-x)e^{-(y-x)} - (1-y+x)e^{-3(y-x)}\right]}{2\left[1+2(y-x)^2\right]e^{-2(y-x)} + 4R_3(y-x)e^{-2(y-x)} - (1+R_3)e^{-4(y-x)} - (1-R_3)}$$

$$= K_1\left[C_3e^{y} + D_3e^{-y}\right]$$

Together with the τ_{rz}-condition, we obtain then the system

$$A_3e^{y} - B_3e^{-y} + C_3(2\mu_3+y)e^{y} + D_3(2\mu_3-y)e^{-y} = 0$$

$$A_3e^{y} + B_3e^{-y} - C_3(1-2\mu_3-y+\Phi_3)e^{y} + D_3(1-2\mu_3+y-\Phi_3)e^{-y} = -\frac{\Phi_2}{\Phi_1}$$

where

$$\Phi_1 = 2(2y-2x+R_3)e^{-2(y-x)} + (1-R_3) - (1+R_3)e^{-4(y-x)}$$

$$\Phi_2 = \frac{2R_2}{R_1}\left[(1+y-x)e^{-(y-x)} - (1-y+x)e^{-3(y-x)}\right]$$

$$\nabla_2 = 2\left[1+2(y-x)^2\right]e^{-2(y-x)} + 4R_3(y-x)e^{-2(y-x)} - (1+R_3)e^{-4(y-x)} - (1-R_3)$$

$$Q_3 = \frac{K_1 . \nabla_2}{Q_1}$$

We notice that for $m = \infty$

$$Q_1 = 1-R_3 \qquad Q_2 = 0 \qquad \nabla_2 = -(1-R_3) \qquad Q_3 = -K_1$$

We write the system in matrix form

$$\left[e^{y} \begin{vmatrix} 1 & 0 & -(1-2\mu_3-y+Q_3) & 0 \\ 1 & 0 & 2\mu_3+y & 0 \end{vmatrix} + e^{-y} \begin{vmatrix} 0 & 1 & 0 & (1-2\mu_3+y-Q_3) \\ 0 & -1 & 0 & 2\mu_3-y \end{vmatrix} \right] \begin{vmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{vmatrix} = \begin{vmatrix} -\frac{Q_2}{Q_1} \\ 0 \end{vmatrix}$$

or

$$\left[e^{y}. M_{41} + e^{-y} M_{42}\right](A_3 \ B_3 \ C_3 \ D_3)^T = \left(-\frac{Q_2}{Q_1} \quad 0\right)^T$$

We replace the matrix $(A_3 \ B_3 \ C_3 \ D_3)^T$ in function of B_4 and C_4

$$\left[e^{y}. M_{41} + e^{-y} M_{42}\right]\left[e^{-2t} M_{5161} + e^{-2x} M_{5162} + e^{-2(t-x)} M_{5261} + M_{5262}\right](B_4 \ D_4)^T$$

$$= 4(1-\mu_3)\left(\frac{Q_2}{Q_1} \quad 0\right)^T$$

We notice that $M_{41} . M_{5261} = 0$

$M_{41} . M_{5262} = 0$

$M_{42} . M_{5161} = 0$

$M_{42} . M_{5162} = 0$

We call

$$M_{41} \cdot M_{5161} = N_1$$
$$M_{41} \cdot M_{5162} = N_2$$
$$M_{42} \cdot M_{5261} = N_3$$
$$M_{42} \cdot M_{5262} = N_4$$

and write

$$\left[e^{-2(r-y)} \cdot N_1 + e^{-(2x-y)} \cdot N_2 + e^{-2(r-x)} \cdot e^{-y} N_3 + e^{-y} N_4 \right] (B_4 \quad D_4)^T$$

$$= 4(1-\mu_3) \left(\frac{\Phi_2}{\Phi_1} \quad 0 \right)^T$$

that we transform into

$$\left[e^{-2(r-y)} N_1 + e^{-2(x-y)} N_2 + e^{-2(r-x)} N_3 + N_4 \right] (B_4 \cdot e^{-y} \quad D_4 e^{-y})^T$$

$$= 4(1-\mu_3) \left(\frac{\Phi_2}{\Phi_1} \quad 0 \right)^T$$

We write

$$e^{-2(r-y)} N_1 + e^{-2(x-y)} N_2 + e^{-2(r-x)} N_3 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$N_4 = \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix}$$

where all the terms a_{ij} converge when $m = \infty$

We devellop the matrix equation

$$(a_{11} + b_{11}) B_4 e^{-y} + (a_{12} + b_{12}) D_4 e^{-y} = 4(1-\mu_3) \frac{\Phi_2}{\Phi_1}$$

$$(a_{21} + b_{21}) B_4 e^{-y} + (a_{22} + b_{22}) D_4 e^{-y} = 0$$

We solve the system

$$B_4 e^{-y} = 4(1-\mu_3)\frac{Q_2}{Q_1} \quad \frac{a_{22}+b_{22}}{\nabla_3}$$

$$D_4 e^{-y} = -4(1-\mu_3)\frac{Q_2}{Q_1} \quad \frac{a_{21}+b_{21}}{\nabla_3}$$

where

$$\nabla_3 = (a_{11}+b_{11})(a_{22}+b_{22}) - (a_{12}+b_{12})(a_{21}+b_{21})$$

$$= (a_{11}+b_{11})a_{22} - (a_{12}+b_{12})a_{21} + b_{11}.b_{22} - b_{12}.b_{21} + a_{11}.b_{22} - a_{12}.b_{21}$$

The term $b_{11}.b_{22} - b_{12}.b_{21}$ contains linear functions of the variables and has to be develloped in close form

$$\begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 1-2\mu_3+y-Q_3 \\ 0 & -1 & 0 & 2\mu_3-y \end{vmatrix} . \begin{vmatrix} 0 & 0 \\ L_1 & L_3+2(L_1-L_2) \\ 0 & 0 \\ 0 & L_2 \end{vmatrix}$$

$$b_{11} = L_1$$

$$b_{12} = L_3+2(L_1-L_2) + L_2(1-2\mu_3+y-Q_3)$$

$$b_{21} = -L_1$$

$$b_{22} = -L_3-2(L_1-L_2) + L_2(2\mu_3-y)$$

$$b_{11}.b_{22} - b_{12}.b_{21} = L_1\Big[-L_3-2(L_1-L_2) + L_2(2\mu_3-y)$$
$$+ L_3 + 2(L_1-L_2) + L_2(1-2\mu_3+y-Q_3)\Big]$$
$$= L_1 L_2 (1-Q_3)$$

The linear functions of the variables have disappeared [illegible] the numerators of $B_4.e^{-y}$ and $D_4.e^{-y}$ tend both to zero, [illegible] factor Q_2, and the denominator tends to a constant

$$\lim_{m\to\infty} Q_1 L_1 L_2 (1-Q_3) = (1+ [illegible]$$

and finally

$$B_4 e^{-x} = B_4 e^{-y} e^{-(x-y)} \quad [illegible]$$

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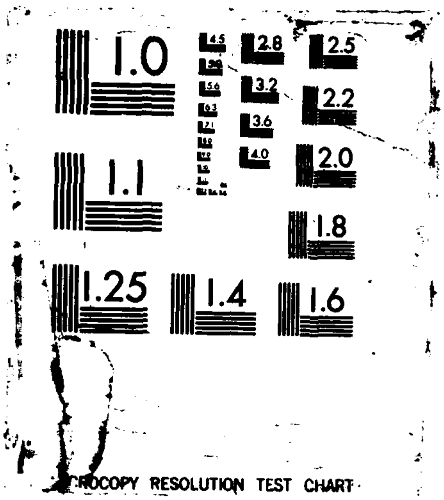
$$
\begin{array}{cccc}
s_2(1+\mu_2) & -s_2(k_2+\mu_2) & s_2 & -s_2 \\
s_2(1+\mu_2)e^{(y-x)} & s_2(k_2+\mu_2)e^{(y-x)} & -s_2e^{(y-x)} & -s_2e^{(y-x)}
\end{array}
$$


1.0
2.8
2.5
3.2
2.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6
CROCOPY RESOLUTION TEST CHART

3. Values of the parameters A_i, D_i.

We express the values of the parameters A_i, D_i in the same way as explained in appendix 1.

3.1. Values of the parameters A_3, B_3, C_3, D_3

The values of the parameters A_3, B_3, C_3 and D_3 are obtained from the relation

$$(A_3 \;\; B_3 \;\; C_3 \;\; D_3)^T = -\frac{1}{4(1-\mu_3)} \cdot \Big[e^{-2t} M_{5161} + e^{-2x} M_{5162} + e^{-2(r-x)} M_{5261} + M_{5262} \Big] (B_4 \;\; D_4)^T$$

The matrices M_{5261} and M_{5262} contain nothing but zeros in their first and third rows, so that we can write

$$(A_3 \;\; 0 \;\; C_3 \;\; 0)^T = -\frac{1}{4(1-\mu_3)} \Big[e^{-2t} M_{5161} + e^{-2x} M_{5162} \Big] (B_4 \;\; D_4)^T$$

$$(A_3 e^{x} \;\; 0 \;\; C_3 e^{x} \;\; 0)^T = -\frac{1}{4(1-\mu_3)} \Big[e^{-(r-x)} \cdot e^{-(r-y)} M_{5161} + e^{-(x-y)} M_{5162} \Big] (B_4 e^{-y} \;\; D_4 e^{-y})^T$$

The matrices M_{5161} and M_{5162} contain nothing but zeros in their second and fourth rows, so that we can write

$$(0 \;\; B_3 \;\; 0 \;\; D_3)^T = -\frac{1}{4(1-\mu_3)} \Big[e^{-2(r-x)} M_{5261} + M_{5262} \Big] (B_4 \;\; D_4)^T$$

$$(0 \;\; B_3 e^{-y} \;\; 0 \;\; D_3 e^{-y})^T = -\frac{1}{4(1-\mu_3)} \Big[e^{-2(r-x)} M_{5261} + M_{5262} \Big] (B_4 e^{-y} \;\; D_4 e^{-y})^T$$

3.2. Values of the parameters A_2, B_2, C_2, D_2.

We have

$$C_2 e^{y} = \frac{[A3]\left[(2y-2x+R_3-1)e^{-2(y-x)} + (1-R_3)\right] + \frac{R_2}{R_1}\left[(1+2y-2x)e^{-(y-x)} - e^{-3(y-x)}\right]}{\nabla_2}$$

$$D_2 e^{-y} = \frac{[A3]\left[(1+2y-2x+R_3)e^{-2(y-x)} - (1+R_3)e^{-4(y-x)}\right] + \frac{R_2}{R_1}\left[e^{-(y-x)} - (1-2y+2x)e^{-3(y-x)}\right]}{\nabla_2}$$

so that

$$D_2 e^{-x} = D_2 e^{-y} . e^{(y-x)}$$

$$= \frac{[A3]\left[(1+2y-2x+R_3)e^{-(y-x)} - (1+R_3)e^{-3(y-x)}\right] + \frac{R_2}{R_1}\left[1 - (1-2y+2x)e^{-2(y-x)}\right]}{\nabla_2}$$

where

$$[A3] = A_3 e^{y} + B_3 e^{-y} - C_3(1-2\mu_3-y)e^{y} + D_3(1-2\mu_3+y)e^{-y}$$

$$= (A_3 e^{z}).e^{-(z-y)} + B_3 e^{-y} - (C_3 e^{z}).(1-2\mu_3-y)e^{-(z-y)} + D_3(1-2\mu_3+y)e^{-y}$$

We have also that

$$2A_2 e^{y} - C_2(1-4\mu_2-2y)e^{y} + D_2 e^{-y} = [A3]$$

so that

$$A_2 e^{y} = \frac{1}{2}\left\{ [A3] + C_2(1-4\mu_2-2y)e^{y} - (D_2 e^{-x}).e^{(y-x)} \right\}$$

and finally we have that

$$2B_2 = -\frac{R_2}{R_1}e^{x} + C_2(1+R_3)e^{2x} - D_2(1-4\mu_2+2x-R_3)$$

so that

$$B_2 e^{-x} = \frac{1}{2}\left[-\frac{R_2}{R_1} + (C_2 e^{y}).(1+R_3)e^{-(y-x)} - (D_2 e^{-x}).(1-4\mu_2+2x-R_3)\right]$$

3.3. Values of the parameters A_1 and C_1.

We have that

$$C_1 e^{x} = \frac{[A2].\left[(2x-1)e^{-2x}+1\right] - (1+2x)e^{-x} + e^{-3x}}{\nabla_1}$$

$$D_1 e^{-x} = \frac{[A2]\left[(1+2x)e^{-2x} - e^{-4x}\right] + (1-2x)\,e^{-3x} - e^{-x}}{\nabla_1}$$

where

$$[A2] = A_2 e^{x} + B_2 e^{-x} - C_2(1-2\mu_2-x)\,e^{x} + D_2(1-2\mu_2+x)\,e^{-x}$$

$$= (A_2 e^{y}).e^{-(y-x)} + B_2 e^{-x} - (C_2 e^{y}).(1-2\mu_2-x)\,e^{-(y-x)} + D_2 e^{-x}.(1-2\mu_2+x)$$

We have also that

$$2A_1 e^{x} - C_1(1-4\mu_1-2x)\,e^{x} + D_1 e^{-x} = [A2]$$

so that

$$A_1 e^{x} = \frac{1}{2}\left\{ [A2] + C_1(1-4\mu_1-2x)\,e^{x} - D_1 e^{-x} \right\}$$

The values of B_1 and D_1 are obtained from the surface conditions

$$A_1 + B_1 - C_1(1-2\mu_1) + D_1(1-2\mu_1) = 1$$

$$A_1 - B_1 + C_1\, 2\mu_1 + D_1\, 2\mu_1 = 0$$

so that

$$B_1 = 2\mu_1 + A_1(1-4\mu_1) + 4\mu_1 C_1 (1-2\mu_1)$$

$$= 2\mu_1 + (A_1 e^{x})(1-4\mu_1)\,e^{-x} + 4\mu_1 (C_1 e^{x})(1-2\mu_1)\,e^{-x}$$

$$D_1 = 1 - 2A_1 + C_1(1-4\mu_1)$$

$$= 1 - 2(A_1 e^{x})\,e^{-x} + (C_1 e^{x})(1-4\mu_1)\,e^{-x}$$

4. Relations for the stresses and the displacements.

The relations for the stresses and displacements are completely the same as those developped in appendix 1, by replacing the parameters A_i, D_i by their adequate values.

Nevertheless, there is a problem in the computation of the vertical displacement: its value at the origin (m = 0) is undeterminated. The relation for the vertical deflection at the surface is given by

$$w = pa\frac{1+\mu_1}{E_1}\int_0^\infty \frac{J_0(mr)\,J_1(ma)}{m}\left[A_1 - B_1 - C_1(2-4\mu_1) - D_1(2-4\mu_1)\right]$$

which, to avoid convergency problems, is transformed into

$$w = -pa\frac{2(1-\mu_1^2)}{E_1}\int_0^\infty \frac{J_0(mr)\,J_1(ma)}{m}\left\{1 - 2\left[A_1 - (1-2\mu_1)C_1\right]\right\}dm$$

The numerators and denominator of A_1 and C_1 are both zero for m = 0. To eliminate the indetermination we should develop A_1 and C_1 in a Taylor series. Altough this is theoretically possible, the required computation is very long and the risks of introducing errors, in doing so, are enormous. Fortunately, we dispose over the fixed bottom condition, which, for m = 0, transforms into

$$A_4 - B_4 - C_4(2-4\mu_4) - D_4(2-4\mu_4) = 0$$

The w-conditions at the other interfaces transform into

$$A_3 - B_3 - C_3(2-4\mu_3) - D_3(2-4\mu_3) = L\left[A_4 - B_4 - C_4(2-4\mu_4) - D_4(2-4\mu_4)\right]$$
$$A_2 - B_2 - C_2(2-4\mu_2) - D_2(2-4\mu_2) = K\left[A_3 - B_3 - C_3(2-4\mu_3) - D_3(2-4\mu_3)\right]$$
$$A_1 - B_1 - C_1(2-4\mu_1) - D_2(2-4\mu_1) = F\left[A_2 - B_2 - C_2(2-4\mu_2) - D_2(2-4\mu_2)\right]$$

so that, for m = 0,

$$A_1 - B_1 - C_1\left[2-4\mu_1\right] - D_1\left[2-4\mu_1\right] = 0$$

and thus

$$\left[A_1 - (1-2\mu_1)C_1\right] = \frac{1}{2}$$

so that the problem is solved without any difficulty.

$$b_{21} b_{12} - b_{11} b_{22} = e^{(y-x)} e^{s_2(y-x)}.$$

APPENDIX 3

ALGEBRAICAL ANALYSIS OF ANISOTROPIC LAYERED SYSTEMS WITH FIXED BOTTOM AND PARTIAL FRICTION CONDITIONS AT THE INTERFACES

SYMBOLS

We write

$$A_i m^2 n_i (1+\mu_i)\, e^{m(H_1+H_2+\dots H_i)} = A_i$$

$$B_i m^2 n_i (1+\mu_i)\, e^{-m(H_1+H_2+\dots H_{i-1})} = B_i$$

$$C_i m^2 n_i s_i (n_i+\mu_i)\, e^{m s_i (H_1+H_2+\dots H_i)} = C_i$$

$$D_i m^2 n_i s_i (n_i+\mu_i)\, e^{-m s_i (H_1+H_2+\dots H_{i-1})} = D_i$$

$$F_w = \frac{E_1}{E_2} \qquad F_u = \lambda_1 \frac{E_1}{E_2}$$

$$K_w = \frac{E_2}{E_3} \qquad K_u = \lambda_2 \frac{E_2}{E_3}$$

$$L_w = \frac{E_3}{E_4} \qquad L_u = \lambda_3 \frac{E_3}{E_4}$$

$$s_i = \left(\frac{n_i-\mu_i^2}{n_i^2-\mu_i^2}\right)^{1/2}$$

$$x = m H_1$$

$$y = m(H_1+H_2)$$

$$z = m(H_1+H_2+H_3)$$

$$t = m(H_1+H_2+H_3+H_4)$$

where H_1, H_2, H_3, H_4 are the thicknesses of the successive layers.
The index 1 applies to the surface layer.
We shall successively analyse a two layered, a three layered and a four layered structure.

Chapter 1. The two layered system.

1. The boundary conditions.

Boundary conditions at the surface (z = 0):

$$\sigma_z: \quad A_1 e^{-x} + B_1 + C_1 e^{-s_1 x} + D_1 = 1$$

$$\tau_{rz}: \quad A_1 e^{-x} - B_1 + s_1 C_1 e^{-s_1 x} - D_1 = 0$$

Boundary conditions at the interface (z = H_1):

$$\sigma_z: \quad A_1 + B_1 e^{-x} + C_1 + D_1 e^{-s_1 x} = A_2 e^{-(y-x)} + B_2 + C_2 e^{-s_2(y-x)} + D_2$$

$$\tau_{rz}: \quad A_1 - B_1 e^{-x} + s_1 C_1 - s_1 D_1 e^{-s_1 x} = A_2 e^{-(y-x)} - B_2 + s_2 C_2 e^{-s_2(y-x)} - s_2 D_2$$

$$w: \quad (1+\mu_1) A_1 - (1+\mu_1) B_1 e^{-x} + s_1 (n_1+\mu_1) C_1 - s_1 (n_1+\mu_1) D_1 e^{-s_1 x} =$$
$$F_w \left[(1+\mu_2) A_2 e^{-(y-x)} - (1+\mu_2) B_2 + s_2 (n_2+\mu_2) C_2 e^{-s_2(y-x)} - s_2 (n_2+\mu_2) D_2 \right]$$

$$u: \quad (n_1+\mu_1) A_1 + (n_1+\mu_1) B_1 e^{-x} + (1+\mu_1) C_1 + (1+\mu_1) D_1 e^{-s_1 x} =$$
$$F_u \left[(n_2+\mu_2) A_2 e^{-(y-x)} + (n_2+\mu_2) B_2 + (1+\mu_2) C_2 e^{-s_2(y-x)} + (1+\mu_2) D_2 \right]$$

Boundary conditions at the bottom (z = $H_1 + H_2$):

$$w: \quad (1+\mu_2) A_2 - (1+\mu_2) B_2 e^{-(y-x)} + s_2 (n_2+\mu_2) C_2 - s_2 (n_2+\mu_2) D_2 e^{-s_2(y-x)} = 0$$

If $s_2 > 1$ $\quad C_2 = 0$

$$A_2 e^{-(y-x)} = B_2 e^{-2(y-x)} + \frac{s_2(n_2+\mu_2)}{(1+\mu_2)} D_2 e^{-(y-x)} e^{-s_2(y-x)}$$

If $s_2 < 1$ $\quad A_2 = 0$

$$C_2 e^{-s_2(y-x)} = \frac{(1+\mu_2)}{s_2(n_2+\mu_2)} B_2 e^{-(y-x)} e^{-s_2(y-x)} + D_2 e^{-2s_2(y-x)}$$

2. Resolution of the system of 6 boundary equations.

We write the conditions at the interface in matrix form

$$(A_1 \;\; B_1 \;\; C_1 \;\; D_1)^T = M_1^{-1} M_2 \, (B_2 \;\; D_2)^T$$

$$M_1^{-1} = \frac{1}{2 s_1 (1-n_1)} \begin{pmatrix} s_1(1+\mu_1) & -s_1(n_1+\mu_1) & s_1 & -s_1 \\ s_1(1+\mu_1)e^{x} & s_1(n_1+\mu_1)e^{x} & -s_1 e^{x} & -s_1 e^{x} \\ -s_1(n_1+\mu_1) & (1+\mu_1) & -1 & s_1 \\ -s_1(n_1+\mu_1)e^{s_1 x} & -(1+\mu_1)e^{s_1 x} & e^{s_1 x} & s_1 e^{s_1 x} \end{pmatrix}$$

If $s_2 > 1$

$$M_2 = \begin{pmatrix} 1 + e^{-2(y-x)} & 1 + \dfrac{s_2(n_2+\mu_2)}{(1+\mu_2)} e^{-(1+s_2)(y-x)} \\ -1 + e^{-2(y-x)} & -s_2 + \dfrac{s_2(n_2+\mu_2)}{(1+\mu_2)} e^{-(1+s_2)(y-x)} \\ F_w (1+\mu_2)\left[-1 + e^{-2(y-x)}\right] & F_w\left[-1 + e^{-(1+s_2)(y-x)}\right] s_2(n_2+\mu_2) \\ F_u (n_2+\mu_2)\left[1 + e^{-2(y-x)}\right] & F_u\left[1 + \dfrac{s_2(n_2+\mu_2)^2}{(1+\mu_2)} e^{-(1+s_2)(y-x)}\right](1+\mu_2) \end{pmatrix}$$

If $s_2 < 1$

$$M_2 = \begin{pmatrix} 1 + \dfrac{1+\mu_2}{s_2(n_2+\mu_2)} e^{-(1+s_2)(y-x)} & 1 + e^{-2s_2(y-x)} \\ -1 + \dfrac{1+\mu_2}{s_2(n_2+\mu_2)} e^{-(1+s_2)(y-x)} & -s_2 + s_2 e^{-2s_2(y-x)} \\ F_w\left[-1 + e^{-(1+s_2)(y-x)}\right](1+\mu_2) & F_w\left[-1 + e^{-2s_2(y-x)}\right] s_2(n_2+\mu_2) \\ F_u\left[1 + \dfrac{(1+\mu_2)^2}{s_2(n_2+\mu_2)^2} e^{-(1+s_2)(y-x)}\right](n_2+\mu_2) & F_u\left[1 + e^{-2s_2(y-x)}\right](1+\mu_2) \end{pmatrix}$$

$$M_1^{-1}.M_2 = \begin{vmatrix} R_{11} & R_{12} \\ R_{21}e^{x} & R_{22}e^{x} \\ R_{31} & R_{32} \\ R_{41}e^{s_1x} & R_{42}e^{s_1x} \end{vmatrix} \cdot \frac{1}{2s_1(1-n_1)}$$

$$A_1 = \frac{1}{2s_1(1-n_1)}\left[R_{11}.B_2 + R_{12}.D_2\right]$$

$$B_1 = \frac{1}{2s_1(1-n_1)}\left[R_{21}.B_2 + R_{22}D_2\right]e^{x}$$

$$C_1 = \frac{1}{2s_1(1-n_1)}\left[R_{31}.B_2 + R_{32}D_2\right]$$

$$D_1 = \frac{1}{2s_1(1-n_1)}\left[R_{41}.B_2 + R_{42}D_2\right]e^{s_1x}$$

We write the surface conditions in matrix form

$$M_0(A_1\ B_1\ C_1\ D_1)^T = (1\quad 0)^T$$

$$M_0.MR\ (B_2\quad D_2)^T = (2s_1(1-n_1)\quad 0)^T$$

$$M_0 = \begin{vmatrix} e^{-x} & 1 & e^{-s_1x} & 1 \\ e^{-x} & -1 & s_1e^{-s_1x} & -s_1 \end{vmatrix}$$

$$M_0.MR = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = MA$$

$$a_{11} = R_{11}e^{-x} + R_{21}e^{x} + R_{31}e^{-s_1x} + R_{41}e^{s_1x}$$

$$a_{12} = R_{12}e^{-x} + R_{22}e^{x} + R_{32}e^{-s_1x} + R_{42}e^{s_1x}$$

$$a_{21} = R_{11}e^{-x} - R_{21}e^{x} + s_1R_{31}e^{-s_1x} - s_1R_{41}e^{s_1x}$$

$$a_{22} = R_{12}e^{-x} - R_{22}e^{x} + s_1R_{32}e^{-s_1x} - s_1R_{42}e^{s_1x}$$

We develop the matrix equations

$$a_{11} . B_2 + a_{12} D_2 = 2s_1(1-n_1)$$

$$a_{21} B_2 + a_{22} D_2 = 0$$

$$B_2 = 2s_1(1-n_1)\frac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}}$$

$$D_2 = -2s_1(1-n_1)\frac{a_{21}}{a_{11}a_{22}-a_{12}a_{21}}$$

$$a_{11}a_{22}-a_{12}a_{21} =$$

$$\begin{aligned} e^{x}e^{s_1x}\Big\{ & R_{11}\left[-2R_{22}e^{-x}e^{-s_1x} - (1-s_1)R_{32}e^{-2x}e^{-2s_1x} - (1+s_1)R_{42}e^{-2x}\right] \\ & + R_{21}\left[2R_{12}e^{-x}e^{-s_1x} + (1+s_1)R_{32}e^{-2s_1x} + (1-s_1)R_{42}\right] \\ & + R_{31}\left[(1-s_1)R_{12}e^{-2x}e^{-2s_1x} - (1+s_1)R_{22}e^{-2s_1x} - 2s_1R_{42}e^{-x}e^{-s_1x}\right] \\ & + R_{41}\left[(1+s_1)R_{12}e^{-2x} - (1-s_1)R_{22} + 2s_1R_{32}e^{-x}e^{-s_1x}\right]\Big\} \end{aligned}$$

$$= e^{x}e^{s_1x} . \nabla$$

$$B_2 = 2s_1(1-n_1)\frac{R_{12}e^{-2x}e^{-s_1x} - R_{22}e^{-s_1x} + s_1R_{32}e^{-x}e^{-2s_1x} - s_1R_{42}e^{-x}}{\nabla}$$

$$D_2 = -2s_1(1-n_1)\frac{R_{11}e^{-2x}e^{-s_1x} - R_{21}e^{-s_1x} + s_1R_{31}e^{-x}e^{-2s_1x} - s_1R_{41}e^{-x}}{\nabla}$$

The numerators contain only negative exponents.
The denominator contains negative exponents and the constant

$$(1-s_1)\left[R_{21}.R_{42} - R_{22}.R_{41}\right]$$

3. Relations for the parameters.

The values of the parameters A_1 and C_1 are deduced from the matrix equations

$$A_1 = \frac{1}{2 s_1 (1 - n_1)} \left[R_{11} B_2 + R_{12} D_2 \right]$$

$$C_1 = \frac{1}{2 s_1 (1 - n_1)} \left[R_{31} B_2 + R_{32} D_2 \right]$$

The values of the parameters B_1 and D_1 are deduced from the surface conditions

$$B_1 = - \frac{s_1}{(1 - s_1)} + \frac{(1 + s_1) A_1 e^{-x} + 2 s_1 C_1 e^{-s_1 x}}{(1 - s_1)}$$

$$D_1 = \frac{1}{(1 - s_1)} - \frac{2 A_1 e^{-x} + (1 + s_1) C_1 e^{-s_1 x}}{(1 - s_1)}$$

Chapter 2. The three layered system.

1. Supplementary boundary conditions.

Boundary conditions at the second interface ($z = H_1 + H_2$):

$$\sigma_z: \; A_2 + B_2 e^{-(y-x)} + C_2 + D_2 e^{-s_2(y-x)} = A_3 e^{-(z-y)} + B_3 + C_3 e^{-s_3(z-y)} + D_3$$

$$\tau_{rz}: \; A_2 - B_2 e^{-(y-x)} + s_2 C_2 - s_2 D_2 e^{-s_2(y-x)} = A_3 e^{-(z-y)} - B_3 + s_3 C_3 e^{-s_3(z-y)} - s_3 D_3$$

$$w: \; (1+\mu_2) A_2 - (1+\mu_2) B_2 e^{-(y-x)} + s_2(n_2+\mu_2) C_2 - s_2(n_2+\mu_2) D_2 e^{-s_2(y-x)} =$$
$$k_w\left[(1+\mu_3) A_3 e^{-(z-y)} - (1+\mu_3) B_3 + s_3(n_3+\mu_3) C_3 e^{-s_3(z-y)} - s_3(n_3+\mu_3) D_3\right]$$

$$u: \; (n_2+\mu_2) A_2 + (n_2+\mu_2) B_2 e^{-(y-x)} + (1+\mu_2) C_2 + (1+\mu_2) D_2 e^{-s_2(y-x)} =$$
$$k_u\left[(n_3+\mu_3) A_3 e^{-(z-y)} + (n_3+\mu_3) B_3 + (1+\mu_3) C_3 e^{-s_3(z-y)} + (1+\mu_3) D_3\right]$$

Boundary conditions at the bottom ($z = H_1 + H_2 + H_3$):

$$w: \; (1+\mu_3) A_3 - (1+\mu_3) B_3 e^{-(z-y)} + s_3(n_3+\mu_3) C_3 - s_3(n_3+\mu_3) D_3 e^{-s_3(z-y)} = 0$$

If $s_3 > 1$ $\quad C_3 = 0$

$$A_3 e^{-(z-y)} = B_3 e^{-2(z-y)} + \frac{s_3(n_3+\mu_3)}{(1+\mu_3)} D_3 e^{-(z-y)} e^{-s_3(z-y)}$$

If $s_3 < 1$ $\quad A_3 = 0$

$$C_3 e^{-s_3(z-y)} = \frac{(1+\mu_3)}{s_3(n_3+\mu_3)} D_3 e^{-(z-y)} e^{-s_3(z-y)} + D_3 e^{-2s_3(z-y)}$$

2. Resolution of the system of 10 boundary equations.

We write the conditions at the second interface in matrix form

$$(A_2 \;\; B_2 \;\; C_2 \;\; D_2)^T = M_3^{-1} . M_4 \; (B_3 \;\; D_3)^T$$

$$M_3^{-1} = \begin{bmatrix} s_2(1+\mu_2) & -s_2(n_2+\mu_2) & s_2 & -s_2 \\ s_2(1+\mu_2)e^{(y-x)} & s_2(n_2+\mu_2)e^{(y-x)} & -s_2 e^{(y-x)} & -s_2 e^{(y-x)} \\ -s_2(n_2+\mu_2) & (1+\mu_2) & -1 & s_2 \\ -s_2(n_2+\mu_2)e^{(y-x)} & -(1+\mu_2)e^{(y-x)} & e^{s_2(y-x)} & s_2 e^{s_2(y-x)} \end{bmatrix} \cdot \frac{1}{2s_2(1-n_2)}$$

If $s_3 > 1$

$$M_4 = \begin{bmatrix} 1 + e^{-2(z-y)} & 1 + \frac{s_3(n_3+\mu_3)}{(1+\mu_3)} e^{-(1+s_3)(z-y)} \\ -1 + e^{-2(z-y)} & -s_3 + \frac{s_3(n_3+\mu_3)}{(1+\mu_3)} e^{-(1+s_3)(z-y)} \\ K_w(1+\mu_3)\left[-1 + e^{-2(z-y)}\right] & K_w s_3(n_3+\mu_3)\left[-1 + e^{-(1+s_3)(z-y)}\right] \\ K_n(n_3+\mu_3)\left[1 + e^{2(z-y)}\right] & K_n(1+\mu_3)\left[1 + \frac{s_3(n_3+\mu_3)^2}{(1+\mu_3)^2} e^{-(1+s_3)(z-y)}\right] \end{bmatrix}$$

If $s_3 < 1$

$$M_4 = \begin{bmatrix} 1 + \frac{1+\mu_3}{s_3(n_3+\mu_3)} e^{-(1+s_3)(z-y)} & 1 + e^{-2s_3(z-y)} \\ -1 + \frac{1+\mu_3}{n_3+\mu_3} e^{-(1+s_3)(z-y)} & -s_3 + s_3 e^{-2s_3(z-y)} \\ K_w(1+\mu_3)\left[-1 + e^{-(1+s_3)(z-y)}\right] & K_w s_3(n_3+\mu_3)\left[-1 + e^{-2s_3(z-y)}\right] \\ K_n(n_3+\mu_3)\left[1 + \frac{(1+\mu_3)^2}{s_3(n_3+\mu_3)^2} e^{-(1+s_3)(z-y)}\right] & K_n(1+\mu_3)\left[1 + e^{-2s_3(z-y)}\right] \end{bmatrix}$$

$$M_3^{-1} \cdot M_4 = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} e^{(y-x)} & Q_{22} e^{(y-x)} \\ Q_{31} & Q_{32} \\ Q_{41} e^{s_2(y-x)} & Q_{42} e^{s_2(y-x)} \end{bmatrix}$$

$$A_2 = \frac{1}{2s_2(1-n_2)}\left[Q_{11}\, B_3 + Q_{12}\, D_3\right]$$

$$B_2 = \frac{1}{2s_2(1-n_2)}\left[Q_{21}\, B_3 + Q_{22}\, D_3\right] e^{(y-x)}$$

$$C_2 = \frac{1}{2s_2(1-n_2)}\left[Q_{31}\, B_3 + Q_{32}\, D_3\right]$$

$$D_2 = \frac{1}{2s_2(1-n_2)}\left[Q_{41}\, B_3 + Q_{42}\, D_3\right] e^{s_2(y-x)}$$

We write the conditions at the first interface in matrix form

$$(A_1\; B_1\; C_1\; D_1)^T = M_1^{-1} . M_2 \;(A_2\; B_2\; C_2\; D_2)^T$$

The value of M_1^{-1} is given in § 1.2

$$M_2 = \begin{matrix} e^{-(y-x)} & 1 & e^{-s_2(y-x)} & 1 \\ e^{-(y-x)} & -1 & s_2 e^{-s_2(y-x)} & -s_2 \\ F_w(1+\mu_2)e^{-(y-x)} & -F_w(1+\mu_2) & F_w s_2(n_2+\mu_2)e^{-s_2(y-x)} & -F_w s_2(n_2+\mu_2) \\ F_n(n_2+\mu_2)e^{-(y-x)} & F_n(n_2+\mu_2) & F_n(1+\mu_2)e^{-s_2(y-x)} & F_n(1+\mu_2) \end{matrix}$$

$$M_1^{-1} . M_2 = \begin{matrix} R_{11} e^{-(y-x)} & R_{12} & R_{13} e^{-s_2(y-x)} & R_{14} \\ R_{21} e^{-(y-x)} . e^{x} & R_{22} e^{x} & R_{23} e^{-s_2(y-x)} . e^{x} & R_{24} e^{x} \\ R_{31} e^{-(y-x)} & R_{32} & R_{33} e^{-s_2(y-x)} & R_{34} \\ R_{41} e^{-(y-x)} e^{s_1 x} & R_{42} e^{s_1 x} & R_{43} e^{-s_2(y-x)} e^{s_1 x} & R_{44} e^{s_1 x} \end{matrix} \quad \frac{1}{2s_1(1-n_1)}$$

$$A_1 = \frac{1}{2s_1(1-n_1)} \left[R_{11} A_2 e^{-(y-x)} + R_{12} B_2 + R_{13} C_2 e^{-s_2(y-x)} + R_{14} D_2 \right]$$

$$B_1 = \frac{1}{2s_1(1-n_1)} \left[R_{21} A_2 e^{-(y-x)} + R_{22} B_2 + R_{23} C_2 e^{-s_2(y-x)} + R_{24} D_2 \right] e^{x}$$

$$C_1 = \frac{1}{2s_1(1-n_1)} \left[R_{31} A_2 e^{-(y-x)} + R_{32} B_2 + R_{33} C_2 e^{-s_2(y-x)} + R_{34} D_2 \right]$$

$$D_1 = \frac{1}{2s_1(1-n_1)} \left[R_{41} A_2 e^{-(y-x)} + R_{42} B_2 + R_{43} C_2 e^{-s_2(y-x)} + R_{44} D_2 \right] e^{s_1 x}$$

$$(A_1 \; B_1 \; C_1 \; D_1)^T = \frac{1}{4 s_1 s_2 (1-n_1)(1-n_2)} \cdot MR.MQ. \; (B_3 \quad D_3)^T$$

$$MR.MQ = \begin{matrix} b_{11} & b_{12} \\ b_{21} e^{x} & b_{22} e^{x} \\ b_{31} & b_{32} \\ b_{41} e^{s_1 x} & b_{42} e^{s_1 x} \end{matrix}$$

$$\begin{aligned}
b_{11} &= R_{11} Q_{11} e^{-(y-x)} + R_{12} Q_{21} e^{(y-x)} + R_{13} Q_{31} e^{-s_2(y-x)} + R_{14} Q_{41} e^{s_2(y-x)} \\
b_{12} &= R_{11} Q_{12} \quad R_{12} Q_{22} \quad R_{13} Q_{32} \quad R_{14} Q_{42} \\
b_{21} &= R_{21} Q_{11} \quad R_{22} Q_{21} \quad R_{23} Q_{31} \quad R_{24} Q_{41} \\
b_{22} &= R_{21} Q_{12} \quad R_{22} Q_{22} \quad R_{23} Q_{32} \quad R_{24} Q_{42} \\
b_{31} &= R_{31} Q_{11} \quad R_{32} Q_{21} \quad R_{33} Q_{31} \quad R_{34} Q_{41} \\
b_{32} &= R_{32} Q_{12} \quad R_{32} Q_{22} \quad R_{33} Q_{32} \quad R_{34} Q_{42} \\
b_{41} &= R_{41} Q_{11} \quad R_{42} Q_{21} \quad R_{43} Q_{31} \quad R_{44} Q_{41} \\
b_{42} &= R_{41} Q_{12} \quad R_{42} Q_{22} \quad R_{43} Q_{32} \quad R_{44} Q_{42}
\end{aligned}$$

We write the surface conditions in matrix form

$$M_0 \,(A_1 \;\; B_1 \;\; C_1 \;\; D_1)^T = (1 \quad 0)^T$$

The value of M_0 is given in § 1.2

$$M_0(A_1 \;\; B_1 \;\; C_1 \;\; D_1)^T = \frac{1}{4 s_1 s_2 (1-n_1)(1-n_2)} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \;(B_3 \quad D_3)^T$$

$$a_{11} = b_{11} e^{-x} + b_{21} e^{x} + b_{31} e^{-s_1 x} + b_{41} e^{s_1 x}$$

$$a_{12} = b_{12} e^{-x} + b_{22} e^{x} + b_{32} e^{-s_1 x} + b_{42} e^{s_1 x}$$

$$a_{21} = b_{11} e^{-x} - b_{21} e^{x} + s_1 b_{31} e^{-s_1 x} - s_1 b_{41} e^{s_1 x}$$

$$a_{22} = b_{12} e^{-x} - b_{22} e^{x} + s_1 b_{32} e^{-s_1 x} - s_1 b_{42} e^{s_1 x}$$

We develop the matrix equation

$$a_{11} B_3 + a_{12} D_3 = 4 s_1 s_2 (1-n_1)(1-n_2)$$

$$a_{21} B_3 + a_{22} D_3 = 0$$

and solve the system

$$B_3 = 4 s_1 s_2 (1-n_1)(1-n_2) \frac{a_{22}}{a_{11} . a_{22} - a_{21} . a_{22}}$$

$$D_3 = -4 s_1 s_2 (1-n_1)(1-n_2) \frac{a_{21}}{a_{11} . a_{22} - a_{21} . a_{22}}$$

$$\nabla = a_{11} . a_{22} - a_{21} . a_{22}$$

We devdop the denominator in closed form

$$a_{11} \cdot a_{22} - a_{21} \cdot a_{12} = e^{x} \cdot e^{s_1 x} \cdot$$

$$\cdot \Big\{ 2\left[b_{21} b_{12} - b_{11} b_{22}\right] e^{-x} e^{-s_1 x} + (1-s_1)\left[b_{31} b_{12} - b_{11} b_{32}\right] e^{-2x} e^{-2s_1 x}$$

$$+ (1+s_1)\left[b_{41} b_{12} - b_{11} b_{42}\right] e^{-2x} + (1+s_1)\left[b_{21} b_{32} - b_{31} b_{22}\right] e^{-2s_1 x}$$

$$+ (1-s_1)\left[b_{21} b_{42} - b_{41} b_{22}\right] + 2s_1\left[b_{41} b_{32} - b_{31} b_{42}\right] e^{-x} e^{-s_1 x} \Big\}$$

$$b_{21} b_{12} - b_{11} b_{22} =$$

$$\left[Q_{11} Q_{22} - Q_{21} Q_{12}\right]\left[R_{21} R_{12} - R_{11} R_{22}\right]$$

$$+ \left[Q_{11} Q_{32} - Q_{31} Q_{12}\right]\left[R_{21} R_{13} - R_{11} R_{23}\right] e^{-(y-x)} e^{-s_2(y-x)}$$

$$+ \left[Q_{11} Q_{42} - Q_{41} Q_{12}\right]\left[R_{21} R_{14} - R_{11} R_{24}\right] e^{-(y-x)} e^{s_2(y-x)}$$

$$+ \left[Q_{21} Q_{32} - Q_{31} Q_{22}\right]\left[R_{22} R_{13} - R_{12} R_{23}\right] e^{(y-x)} e^{-s_2(y-x)}$$

$$+ \left[Q_{21} Q_{42} - Q_{41} Q_{22}\right]\left[R_{22} R_{14} - R_{12} R_{24}\right] e^{(y-x)} e^{s_2(y-x)}$$

$$+ \left[Q_{31} Q_{42} - Q_{41} Q_{32}\right]\left[R_{23} R_{14} - R_{13} R_{24}\right]$$

We write

$$\left[Q_{11} Q_{22} - Q_{21} Q_{12}\right] e^{-(y-x)} e^{-s_2(y-x)} = q_{12}$$

$$\left[Q_{11} Q_{32} - Q_{31} Q_{12}\right] e^{-2(y-x)} e^{-2s_2(y-x)} = q_{13}$$

$$\left[Q_{11} Q_{42} - Q_{41} Q_{12}\right] e^{-2(y-x)} = q_{14}$$

$$\left[Q_{21} Q_{32} - Q_{31} Q_{22}\right] e^{-2s_2(y-x)} = q_{23}$$

$$\left[Q_{21} Q_{42} - Q_{41} Q_{22}\right] = q_{24}$$

$$\left[Q_{31} Q_{42} - Q_{41} Q_{32}\right] e^{-(y-x)} e^{-s_2(y-x)} = q_{34}$$

$$b_{21} b_{12} - b_{11} b_{22} = e^{(y-x)} \cdot e^{s_2(y-x)} \cdot$$

$$\Big\{ q_{12}\left[R_{21} R_{12} - R_{11} R_{22}\right] + q_{13}\left[R_{21} R_{13} - R_{11} R_{23}\right]$$
$$+ q_{14}\left[R_{21} R_{14} - R_{11} R_{24}\right] + q_{23}\left[R_{22} R_{13} - R_{12} R_{23}\right]$$
$$+ q_{24}\left[R_{22} R_{14} - R_{12} R_{24}\right] + q_{34}\left[R_{23} R_{14} - R_{13} R_{24}\right] \Big\}$$

$$= e^{(y-x)} e^{s_2(y-x)} Q R_1$$

$$b_{31} b_{12} - b_{11} b_{32} = e^{(y-x)} e^{s_2(y-x)} \cdot$$

$$\Big\{ q_{12}\left[R_{31} R_{12} - R_{11} R_{32}\right] + q_{13}\left[R_{31} R_{13} - R_{11} R_{33}\right]$$
$$+ q_{14}\left[R_{31} R_{14} - R_{11} R_{34}\right] + q_{23}\left[R_{32} R_{13} - R_{12} R_{33}\right]$$
$$+ q_{24}\left[R_{32} R_{14} - R_{12} R_{34}\right] + q_{34}\left[R_{33} R_{14} - R_{13} R_{34}\right] \Big\}$$

$$= e^{(y-x)} e^{s_2(y-x)} Q R_2$$

$$b_{41} \cdot b_{12} - b_{11} b_{42} = e^{(y-x)} e^{s_2(y-x)} \cdot$$

$$\Big\{ q_{12}\left[R_{41} R_{12} - R_{11} R_{42}\right] + q_{13}\left[R_{41} R_{13} - R_{11} R_{43}\right]$$
$$+ q_{14}\left[R_{41} R_{14} - R_{11} R_{44}\right] + q_{23}\left[R_{42} R_{13} - R_{12} R_{43}\right]$$
$$+ q_{24}\left[R_{42} R_{14} - R_{12} R_{44}\right] + q_{34}\left[R_{43} R_{14} - R_{13} R_{44}\right] \Big\}$$

$$= e^{(y-x)} e^{s_2(y-x)} Q R_3$$

$$b_{21} \cdot b_{32} - b_{31} b_{22} = e^{(y-x)} e^{s_2(y-x)} \cdot$$

$$\Big\{ q_{12}\left[R_{21} \cdot R_{32} - R_{31} R_{22}\right] + q_{13}\left[R_{21} R_{33} - R_{31} R_{23}\right]$$
$$+ q_{14}\left[R_{21} R_{34} - R_{31} R_{24}\right] + q_{23}\left[R_{22} R_{33} - R_{32} R_{23}\right]$$
$$+ q_{24}\left[R_{22} R_{34} - R_{32} R_{24}\right] + q_{34}\left[R_{23} R_{34} - R_{33} R_{24}\right] \Big\}$$

$$= e^{(y-x)} e^{s_2(y-x)} Q R_4$$

$$b_{21} b_{42} - b_{41} b_{22} = e^{(y-x)} e^{s_2(y-x)} .$$

$$\Big\{ q_{12} \left[R_{21} R_{42} - R_{41} R_{22} \right] + q_{13} \left[R_{21} R_{43} - R_{41} R_{23} \right]$$

$$+ q_{14} \left[R_{21} R_{44} - R_{41} R_{24} \right] + q_{23} \left[R_{22} R_{43} - R_{42} R_{23} \right]$$

$$+ q_{24} \left[R_{22} R_{44} - R_{42} R_{24} \right] + q_{34} \left[R_{23} R_{44} - R_{43} R_{24} \right] \Big\}$$

$$= e^{(y-x)} e^{s_2(y-x)} Q R_5$$

$$b_{41} b_{32} - b_{31} b_{42} = e^{(y-x)} e^{s_2(y-x)} .$$

$$\Big\{ q_{12} \left[R_{41} R_{32} - R_{31} R_{42} \right] + q_{13} \left[R_{41} R_{33} - R_{31} R_{43} \right]$$

$$+ q_{14} \left[R_{41} R_{34} - R_{31} R_{44} \right] + q_{23} \left[R_{42} R_{33} - R_{32} R_{43} \right]$$

$$+ q_{24} \left[R_{42} R_{34} - R_{32} R_{44} \right] + q_{34} \left[R_{43} R_{34} - R_{33} R_{44} \right] \Big\}$$

$$= e^{(y-x)} e^{s_2(y-x)} Q R_6$$

$$\nabla = e^{x} e^{s_1 x} e^{(y-x)} e^{s_2(y-x)} .$$

$$\Big[2 Q R_1 e^{-x} e^{-s_1 x} + (1-s_1) Q R_2 e^{-2x} e^{-2s_1 x} + (1+s_1) Q R_3 e^{-2x}$$

$$+ (1+s_1) Q R_4 e^{-2s_1 x} + (1-s_1) Q R_5 + 2 s_1 Q R_6 e^{-x} e^{-s_1 x} \Big]$$

Between the brackets, the denominator has next constant term

$$(1-s_1) Q R_5 = (1-s_1) \left[Q_{21} Q_{42} - Q_{41} Q_{22} \right] \left[R_{22} R_{44} - R_{42} R_{24} \right]$$

$$\nabla = e^{x} e^{s_1 x} e^{(y-x)} e^{s_2(y-x)} \nabla'$$

We write then $\qquad b_{ij} = e^{(y-x)} e^{s_2(y-x)} b'_{ij}$

$$b'_{11} = R_{11} Q_{11} e^{-2(y-x)} e^{-s_2(y-x)} + R_{12} Q_{21} e^{-s_1(y-x)} + R_{13} Q_{31} e^{-(y-x)} e^{-2s_2(y-x)} + R_{14} Q_{41} e^{-(y-x)}$$

$$\begin{array}{llll}
b'_{12} = R_{11} Q_{12} & R_{12} Q_{22} & R_{13} Q_{32} & R_{14} Q_{42} \\
b'_{21} = R_{21} Q_{11} & R_{22} Q_{21} & R_{23} Q_{31} & R_{24} Q_{41} \\
b'_{22} = R_{21} Q_{12} & R_{22} Q_{22} & R_{23} Q_{32} & R_{24} Q_{42} \\
b'_{31} = R_{31} Q_{11} & R_{32} Q_{21} & R_{33} Q_{31} & R_{34} Q_{41} \\
b'_{32} = R_{31} Q_{12} & R_{32} Q_{22} & R_{33} Q_{32} & R_{34} Q_{42} \\
b'_{41} = R_{41} Q_{11} & R_{42} Q_{21} & R_{43} Q_{31} & R_{44} Q_{41} \\
b'_{42} = R_{41} Q_{12} & R_{42} Q_{22} & R_{43} Q_{31} & R_{44} Q_{42}
\end{array}$$

$$a_{ij} = e^{x} e^{s_1 x} e^{(y-x)} e^{s_2(y-x)} a'_{ij}$$

$$a'_{21} = b'_{11} e^{-2x} e^{-s_1 x} - b'_{21} e^{-s_1 x} + s_1 b'_{31} e^{-x} e^{3s_1 x} - s_1 b'_{41} e^{-x}$$

$$a'_{22} = b'_{12} e^{-2x} e^{-s_1 x} - b'_{22} e^{-s_1 x} + s_1 b'_{32} e^{-x} e^{-2s_1 x} - s_1 b'_{42} e^{-x}$$

$$B_3 = 4 s_1 s_2 (1-n_1)(1-n_2) \frac{a_{22}}{\nabla}$$

$$= 4 s_1 s_2 (1-n_1)(1-n_2) \frac{e^{x} e^{s_1 x} e^{(y-x)} e^{s_2(y-x)} a'_{22}}{e^{x} e^{s_1 x} e^{(y-x)} e^{s_2(y-x)} \nabla'}$$

$$= 4 s_1 s_2 (1-n_1)(1-n_2) \frac{a'_{22}}{\nabla'}$$

$$D_3 = 4 s_1 s_2 (1-n_1)(1-n_2) \frac{a'_{21}}{\nabla'}$$

The numerators contain only negative exponents; the denominator contains a constant term and negative exponents.

3. Relations for the parameters.

The parameters A_2 and C_2 can immediately be deduced from B_3 and D_3

$$A_2 = \frac{1}{2s_2(1-n_2)} \left[Q_{11} B_3 + Q_{12} D_3 \right]$$

$$C_2 = \frac{1}{2s_2(1-n_2)} \left[Q_{31} B_3 + Q_{32} D_3 \right]$$

The same relations for B_2 and D_2 contain positive exponents which have to be eliminated.

$$B_2 = \frac{1}{2s_2(1-n_2)} \left[Q_{21} B_3 + Q_{22} D_3 \right] e^{(y-x)}$$

$$= \frac{1}{2s_2(1-n_2)} \quad \frac{4 s_1 s_2 (1-n_1)(1-n_2)}{\nabla'} \cdot e^{(y-x)} .$$

$$\left\{ Q_{21} \left[b'_{12} e^{-2x} e^{-s_1 x} - b'_{22} e^{-s_1 x} + s_1 b'_{32} e^{-x} e^{-2s_1 x} - s_1 b'_{42} e^{-x} \right] \right.$$
$$\left. - Q_{22} \left[b'_{11} e^{-2x} e^{-s_1 x} - b'_{21} e^{-s_1 x} + s_1 b'_{31} e^{-x} e^{-2s_1 x} - s_1 b'_{41} e^{-x} \right] \right\}$$

$$B_2 = \frac{2 s_1 (1-n_1)}{\nabla'} \cdot$$
$$\left\{ \left[-R_{11} q_{12} + R_{13} q_{23} + R_{14} q_{24} \right] e^{-2x} e^{-s_1 x} \right.$$
$$+ \left[R_{21} q_{12} - R_{23} q_{23} - R_{24} q_{24} \right] e^{-s_1 x}$$
$$+ s_1 \left[-R_{31} q_{12} + R_{33} q_{23} + R_{34} q_{24} \right] e^{-x} e^{-s_1 x}$$
$$\left. + s_1 \left[R_{41} q_{12} - R_{43} q_{23} - R_{44} q_{24} \right] e^{-x} \right\}$$

The factors R_{i2} with the negative exponent $-s_2(y - x)$, which does not eliminate against the positive exponent $(y - x)$, have disappeared.

We obtain in the same way

$$D_2 = \frac{2s_1(1-n_1)}{\nabla'} \cdot$$

$$\Big\{ -\left[R_{11} q_{14} + R_{12} q_{24} + R_{13} q_{34}\right] e^{-x} e^{-s_1 x}$$

$$+ \left[R_{21} q_{14} + R_{22} q_{24} + R_{23} q_{34}\right] e^{-s_1 x}$$

$$- s_1 \left[R_{31} q_{14} + R_{32} q_{24} + R_{33} q_{34}\right] e^{-x} e^{-2s_1 x}$$

$$+ s_1 \left[R_{41} q_{14} + R_{42} q_{24} + R_{43} q_{34}\right] e^{-x} \Big\}$$

The parameters A_1 and C_1 are obtained from the matrix relations

$$A_1 = \frac{1}{2s_1(1-n_1)} \left[R_{11} A_2 e^{-(y-x)} + R_{12} B_2 + R_{13} e^{-s_2(y-x)} C_2 + R_{14} D_2\right]$$

$$C_1 = \frac{1}{2s_1(1-n_1)} \left[R_{31} A_2 e^{-(y-x)} + R_{32} B_2 + R_{33} e^{-s_2(y-x)} C_2 + R_{34} D_2\right]$$

The values of B_1 and D_1 are obtained from the surface conditions

$$B_1 = \frac{1}{s_1 - 1} \left[s_1 - (1+s_1) A_1 e^{-x} - 2s_1 C_1 e^{-s_1 x}\right]$$

$$D_1 = -\frac{1}{s_1 - 1} \left[1 - 2A_1 e^{-x} - (1+s_1) C_1 e^{-s_1 x}\right]$$

Chapter 3. The four layered structure.

1. Supplementary boundary conditions.

Boundary conditions at the third interface ($z = H_1 + H_2 + H_3$):

$$\sigma_z: \quad A_3 + B_3 e^{-(z-y)} + C_3 + D_3 e^{-s_3(z-y)} = A_4 e^{-(t-x)} + B_4 + C_4 e^{-s_4(t-x)} + D_4$$

$$\tau_{rz}: \quad A_3 - B_3 e^{-(z-y)} + s_3 C_3 - s_3 D_3 e^{-s_3(z-y)} = A_4 e^{-(t-x)} - B_4 + s_4 C_4 e^{-s_4(t-x)} - s_4 D_4$$

$$w: \quad (1+\mu_3) A_3 - (1+\mu_3) B_3 e^{-(z-y)} + s_3 (n_3+\mu_3) C_3 - s_3 (n_3+\mu_3) D_3 e^{-s_3(z-y)} = L_w \left[(1+\mu_4) A_4 e^{-(t-x)} - (1+\mu_4) B_4 + s_4 (n_4+\mu_4) C_4 e^{-s_4(t-x)} - s_4 (n_4+\mu_4) D_4 \right]$$

$$u: \quad (n_3+\mu_3) A_3 + (n_3+\mu_3) B_3 e^{-(z-y)} + (1+\mu_3) C_3 + (1+\mu_3) D_3 e^{-s_3(z-y)} = L_u \left[(n_4+\mu_4) A_4 e^{-(t-x)} + (n_4+\mu_4) B_4 + (1+\mu_4) C_4 e^{-s_4(t-x)} + (1+\mu_4) D_4 \right]$$

Boundary conditions at the bottom ($z = H_1 + H_2 + H_3 + H_4$):

$W = 0$

$$(1+\mu_4) A_4 - (1+\mu_4) B_4 e^{-(t-x)} + s_4 (n_4+\mu_4) C_4 - s_4 (n_4+\mu_4) D_4 e^{-s_4(t-x)} = 0$$

If $s_4 > 1$ $\quad C_4 = 0$

$$A_4 e^{-(t-x)} = B_4 e^{-2(t-x)} + \frac{s_4 (n_4+\mu_4)}{(1+\mu_4)} D_4 e^{-(t-x)} e^{-s_4(t-x)}$$

If $s_4 < 1$ $\quad A_4 = 0$

$$C_4 e^{-s_4(t-x)} = \frac{(1+\mu_4)}{s_4 (n_4+\mu_4)} B_4 e^{-(t-x)} e^{-s_4(t-x)} + D_4 e^{-2 s_4 (t-x)}$$

2. Expression of the boundary conditions in matrixform.

2.1. At the third interface.

In the equations at the third interface, A_4, or C_4, is replaced by its value obtained from the fixed bottom condition.

We write the conditions at the third interface in matrixform

$$M_5 \left(A_3\, B_3\, C_3\, D_3\right)^T = M_6 \left(B_4\, D_4\right)^T$$

We invert M_5

$$\left(A_3\, B_3\, C_3\, D_3\right)^T = M_5^{-1} \cdot M_6 \left(B_4\, D_4\right)^T$$

$$M_5^{-1} = \frac{1}{2s_3(1-n_3)} \begin{pmatrix} s_3(1+\mu_3) & -s_3(n_3+\mu_3) & s_3 & -s_3 \\ s_3(1+\mu_3)e^{(z-y)} & s_3(n_3+\mu_3)e^{(z-y)} & -s_3e^{(z-y)} & -s_3e^{(z-y)} \\ -s_3(n_3+\mu_3) & (1+\mu_3) & -1 & s_3 \\ -s_3(n_3+\mu_3)e^{s_3(z-y)} & -(1+\mu_3)e^{s_3(z-y)} & e^{s_3(z-y)} & s_3e^{s_3(z-y)} \end{pmatrix}$$

If $s_4 > 1$

$$M_6 = \begin{pmatrix} 1+e^{-2(h-z)} & 1 + \dfrac{s_4(n_4+\mu_4)}{(1+\mu_4)} e^{-(1+s_4)(h-z)} \\ -1+e^{-2(h-z)} & -s_4 + \dfrac{s_4(n_4+\mu_4)}{(1+\mu_4)} e^{-(1+s_4)(h-z)} \\ L_1(1+\mu_4)\left[-1+e^{-2(h-z)}\right] & L_1 s_4(n_4+\mu_4)\left[-1+e^{-(1+s_4)(h-z)}\right] \\ L_2(n_4+\mu_4)\left[1+e^{-2(h-z)}\right] & L_2(1+\mu_4)\left[1+\dfrac{s_4(n_4+\mu_4)^2}{(1+\mu_4)^2} e^{-(1+s_4)(h-z)}\right] \end{pmatrix}$$

If $s_4 < 1$

$$M_6 \begin{pmatrix} 1 + \dfrac{1+\mu_4}{s_4(n_4+\mu_4)} e^{-(1+s_4)(h-z)} & 1+e^{-2s_4(h-z)} \\ -1 + \dfrac{1+\mu_4}{(n_4+\mu_4)} e^{-(1+s_4)(h-z)} & -s_4 + e^{-2s_4(h-z)} \cdot s_4 \\ L_1(1+\mu_4)\left[-1+e^{-(1+s_4)(h-z)}\right] & L_1 s_4(n_4+\mu_4)\left[-1+e^{-2s_4(h-z)}\right] \\ L_2(n_4+\mu_4)\left[1+\dfrac{(1+\mu_4)^2}{s_4(n_4+\mu_4)^2} \cdot e^{-(1+s_4)(h-z)}\right] & L_2(1+\mu_4)\left[1+e^{-2s_4(h-z)}\right] \end{pmatrix}$$

We write A_3, B_3, C_3 and D_3 in function of B_4 and D_4.

$$A_3 = \frac{\sum M_5(1i).M_6(i1).B_4 + \sum M_5(1i).M_6(i2)D_4}{2s_3(1-n_3)}$$

$$B_3 = \frac{\sum M_5(2i).M_6(i1).B_4 + \sum M_5(2i).M_6(i2)D_4}{2s_3(1-n_3)}.e^{(z-y)}$$

$$C_3 = \frac{\sum M_5(3i).M_6(i1).B_4 + \sum M_5(3i).M_6(i2)D_4}{2s_3(1-n_3)}$$

$$D_3 = \frac{\sum M_5(4i).M_6(i1).B_4 + \sum M_5(4i).M_6(i2)D_4}{2s_3(1-n_3)}e^{s_3(z-y)}$$

where $M_5(i,j)$ are the constants in M_5^{-1}.

We write

$$P_{j1} = \sum M_5(ji)M_6(i1)$$

$$P_{j2} = \sum M_5(ji)M_6(i2)$$

so that

$$\begin{pmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{pmatrix} = \frac{1}{2s_3(1-n_3)} \begin{pmatrix} P_{11} & P_{12} \\ P_{21}e^{(z-y)} & P_{22}e^{(z-y)} \\ P_{31} & P_{32} \\ P_{41}e^{s_3(z-y)} & P_{42}e^{s_3(z-y)} \end{pmatrix} . \begin{vmatrix} B_4 \\ D_4 \end{vmatrix}$$

2.2. At the second interface.

We write the conditions at the second interface in matrixform.

$$M_3(A_2\,B_2\,C_2\,D_2)^T = M_4(A_3\,B_3\,C_3\,D_3)^T$$

We invert M_3

$$(A_2\,B_2\,C_2\,D_2)^T = M_3^{-1}.M_4(A_3\,B_3\,C_3\,D_3)^T$$

$$M_3^{-1} = \frac{1}{2s_2(1-n_2)} \begin{pmatrix} s_2(1+\mu_2) & -s_2(n_2+\mu_2) & s_2 & -s_2 \\ s_2(1+\mu_2)e^{(y-x)} & s_2(n_2+\mu_2)e^{(y-x)} & -s_2 e^{(y-x)} & -s_2 e^{(y-y)} \\ -s_2(n_2+\mu_2) & (1+\mu_2) & -1 & s_2 \\ -s_2(n_2+\mu_2)e^{s_2(y-x)} & -(1+\mu_2)e^{s_2(y-x)} & e^{s_2(y-x)} & s_2 e^{s_2(y-x)} \end{pmatrix}$$

$$M_4 = \begin{pmatrix} e^{-(x-y)} & 1 & e^{-s_3(x-y)} & 1 \\ e^{-(x-y)} & -1 & s_3 e^{-s_3(x-y)} & -s_3 \\ k_1(1+\mu_3)e^{-(x-y)} & -k_1(1+\mu_3) & k_1 s_3(n_3+\mu_3)e^{-s_3(x-y)} & -k_1 s_3(n_3+\mu_3) \\ k_2(n_3+\mu_3)e^{-(x-y)} & k_2(n_3+\mu_3) & k_2(1+\mu_3)e^{-s_3(x-y)} & k_2(1+\mu_3) \end{pmatrix}$$

We write A_2, B_2, C_2 and D_2 in function of A_3, B_3, C_3 and D_3.

$$A_2 = \frac{\sum M_3(1i).M_4(i1)\,e^{-(x-y)}.A_3 + \sum M_3(1i).M_4(i2).B_3}{2s_2(1-n_2)}$$
$$+ \frac{\sum M_3(1i).M_4(i3)\,e^{-s_3(x-y)}C_3 + \sum M_3(1i).M_4(i4).D_3}{2s_2(1-n_2)}$$

$$B_2 = \frac{\sum M_3(2i)\,M_4(i1)e^{-(x-y)}A_3 + \sum M_3(2i).M_4(i2).B_3}{2s_2(1-n_2)}.e^{(y-x)}$$
$$+ \frac{\sum M_3(2i).M_4(i3)e^{-s_3(x-y)}.C_3 + \sum M_3(2i).M_4(i4)\,D_3}{2s_2(1-n_2)}.e^{(y-x)}$$

$$C_2 = \frac{\sum M_3(3i).M_4(i1)e^{-(x-y)}A_3 + \sum M_3(3i).M_4(i2)\,B_3}{2s_2(1-n_2)}$$
$$+ \frac{\sum M_3(3i).M_4(i3)e^{-s_3(x-y)}C_3 + \sum M_3(3i).M_4(i4)\,D_3}{2s_2(1-n_2)}$$

$$D_2 = \frac{\sum M_3(4i).M_4(i1)e^{-(x-y)}A_3 + \sum M_3(4i).M_4(i2)\,B_3}{2s_2(1-n_2)}\,e^{s_2(y-x)}$$
$$+ \frac{\sum M_3(4i).M_4(i3)e^{-s_3(x-y)}C_3 + \sum M_3(4i.).M_4(i2)\,D_3}{2s_2(1-n_2)}\,e^{s_2(y-x)}$$

where $M_3(i,j)$ and $M_4(i,j)$ are the constants in M_3^{-1} and M_4.

We write

$$Q_{jl} = \sum M_3(ji) \cdot M_4(il)$$

so that

$$\begin{pmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{pmatrix} = \frac{1}{2s_2(1-n_2)} \begin{pmatrix} Q_{11}e^{-(z-y)} & Q_{12} & Q_{13}e^{-s_3(z-y)} & Q_{14} \\ Q_{21}e^{-(z-y)}e^{(y-x)} & Q_{22}e^{(y-x)} & Q_{23}e^{-s_3(z-y)}e^{(y-x)} & Q_{24}e^{(y-x)} \\ Q_{31}e^{-(z-y)} & Q_{32} & Q_{33}e^{-s_3(z-y)} & Q_{34} \\ Q_{41}e^{-(z-y)}e^{s_2(y-x)} & Q_{42}e^{s_2(y-x)} & Q_{43}e^{-s_3(z-y)}e^{s_2(y-x)} & Q_{44}e^{s_2(y-x)} \end{pmatrix} \cdot \begin{pmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{pmatrix}$$

2.3. At the first interface.

We write the conditions at the first interface in matrixform.

$$M_1 (A_1 B_1 C_1 D_1)^T = M_2 (A_2 B_2 C_2 D_2)^T$$

We invert M_1

$$(A_1 B_1 C_1 D_1)^T = M_1^{-1} \cdot M_2 (A_2 B_2 C_2 D_2)^T$$

$$M_1^{-1} = \frac{1}{2s_1(1-n_1)} \begin{pmatrix} s_1(1+\mu_1) & -s_1(n_1+\mu_1) & s_1 & -s_1 \\ s_1(1+\mu_1)e^{x} & s_1(n_1+\mu_1)e^{x} & -s_1e^{x} & -s_1e^{x} \\ -s_1(n_1+\mu_1) & (1+\mu_1) & -1 & s_1 \\ -s_1(n_1+\mu_1)e^{s_1x} & -(1+\mu_1)e^{s_1x} & e^{s_1x} & s_1e^{s_1x} \end{pmatrix}$$

$$M_2 = \begin{pmatrix} e^{-(y-x)} & 1 & e^{-s_2(y-x)} & 1 \\ e^{-(y-x)} & -1 & s_2e^{-s_2(y-x)} & -s_2 \\ F_1(1+\mu_2)e^{-(y-x)} & -F_1(1+\mu_2) & F_1s_2(n_2+\mu_2)e^{-s_2(y-x)} & -F_1s_2(n_2+\mu_2) \\ F_2(n_2+\mu_2)e^{-(y-x)} & F_2(n_2+\mu_2) & F_2(1+\mu_2)e^{-s_2(y-x)} & F_2(1+\mu_2) \end{pmatrix}$$

We write A_1, B_1, C_1 and D_1 in function of A_2, B_2, C_2 and D_2

$$A_1 = \frac{\sum M_1(1i)\, M_2(i1)\, e^{-(y-x)} A_2 + \sum M_1(1i) \cdot M_2(i2)\, B_2}{2s_1(1-n_1)}$$

$$+ \frac{\sum M_1(1i)\, M_2(i3)\, e^{-s_2(y-x)} C_2 + \sum M_1(1i) \cdot M_2(i4)\, D_2}{2s_1(1-n_1)}$$

$$B_1 = \frac{\sum M_1(2i) \cdot M_2(i1)\, e^{-(y-x)} A_2 + \sum M_1(2i) \cdot M_2(i2)\, B_2}{2s_1(1-n_1)} \cdot e^{x}$$

$$+ \frac{\sum M_1(2i) \cdot M_2(i3)\, e^{-s_2(y-x)} C_2 + \sum M_1(2i) \cdot M_2(i4)\, D_2}{2s_1(1-n_1)} \cdot e^{x}$$

$$C_1 = \frac{\sum M_1(3i) \cdot M_2(i1)\, e^{-(y-x)} A_2 + \sum M_1(3i) \cdot M_2(i2)\, B_2}{2s_1(1-n_1)}$$

$$+ \frac{\sum M_1(3i) \cdot M_2(i3)\, e^{-s_2(y-x)} C_2 + \sum M_1(3i) \cdot M_2(i4)\, D_2}{2s_1(1-n_1)}$$

$$D_1 = \frac{\sum M_1(4i) \cdot M_2(i1)\, e^{-(y-x)} A_2 + \sum M_1(4i) \cdot M_2(i2)\, B_2}{2s_1(1-n_1)} \cdot e^{s_1 x}$$

$$+ \frac{\sum M_1(4i) \cdot M_2(i3)\, e^{-s_2(y-x)} C_2 + \sum M_1(4i) \cdot M_2(i4)\, D_2}{2s_1(1-n_1)} \cdot e^{s_1 x}$$

where $M_1(i,j)$ and $M_2(i,j)$ are the constants in M_1^{-1} and M_2.

We write

$$R_{j\ell} = M_1(ji) \cdot M_2(i\ell)$$

so that

$$\begin{pmatrix} A_1 \\ B_1 \\ C_1 \\ D_1 \end{pmatrix} = \frac{1}{2s_1(1-n_1)} \begin{pmatrix} R_{11} e^{-(y-x)} & R_{12} & R_{13} e^{-s_2(y-x)} & R_{14} \\ R_{21} e^{-(y-x)} \cdot e^{x} & R_{22} e^{x} & R_{23} e^{-s_2(y-x)} \cdot e^{x} & R_{24} e^{x} \\ R_{31} \cdot e^{-(y-x)} & R_{32} & R_{33} e^{-s_2(y-x)} & R_{34} \\ R_{41} e^{-(y-x)} \cdot e^{s_1 x} & R_{42} e^{s_1 x} & R_{43}\, e^{-s_2(y-x)} \cdot e^{s_1 x} & R_{44} e^{s_1 x} \end{pmatrix} \cdot \begin{pmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{pmatrix}$$

3. Resolution of the system of boundary conditions.

We write the conditions at the surface in matrixform

$$\begin{vmatrix} e^{-x} & 1 & e^{-s_1 x} & 1 \\ e^{-x} & -1 & s_1 e^{-s_1 x} & -s_1 \end{vmatrix} \cdot (A_1 B_1 C_1 D_1)^T = \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$

We have then following system of matrix equations

$$MI \cdot (A_1 B_1 C_1 D_1)^T = (1 \quad 0)^T$$

$$(A_1 B_1 C_1 D_1)^T = \frac{1}{2 s_1 (1-n_1)} MR \, (A_2 B_2 C_2 D_2)^T$$

$$(A_2 B_2 C_2 D_2)^T = \frac{1}{2 s_2 (1-n_2)} MQ \, (A_3 B_3 C_3 D_3)^T$$

$$(A_3 B_3 C_3 D_3)^T = \frac{1}{2 s_3 (1-n_3)} \cdot MP \, (B_4 \ D_4)^T$$

so that

$$\frac{1}{8 s_1 s_2 s_3 (1-n_1)(1-n_2)(1-n_3)} \cdot MI.MR.MQ.MP (B_4 \ D_4)^T = (1 \quad 0)^T$$

$$MI.MR = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{34} \end{vmatrix}$$

$$MQ.MP = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix}$$

$$MI.MR.MQ.MP = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix}$$

$$B_4 = 8 s_1 s_2 s_3 (1-h_1)(1-h_2)(1-h_3) \cdot \frac{c_{22}}{c_{11} \cdot c_{22} - c_{12} \cdot c_{21}}$$

$$D_4 = -8 s_1 s_2 s_3 (1-h_1)(1-h_2)(1-h_3) \cdot \frac{c_{21}}{c_{11} \cdot c_{22} - c_{12} \cdot c_{21}}$$

3.1. Determination of $c_{11}.c_{22} - c_{21}.c_{12}$

The expression of the denominator $c_{11}.c_{22} - c_{21}.c_{12}$ must be established in complete closeform altough regarding the exponentials.

$$c_{11} = a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} + a_{14} \cdot b_{41}$$

$$c_{12} = a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} + a_{14} \cdot b_{42}$$

$$c_{21} = a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} + a_{24} \cdot b_{41}$$

$$c_{22} = a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} + a_{24} \cdot b_{42}$$

$$\begin{aligned}
c_{11} \cdot c_{22} - c_{12} \cdot c_{21} = \\
& (a_{11} \cdot a_{22} - a_{12} \cdot a_{21}) \cdot (b_{11} \cdot b_{22} - b_{12} \cdot b_{21}) \\
& + (a_{11} \cdot a_{23} - a_{13} \cdot a_{21}) \cdot (b_{11} \cdot b_{32} - b_{12} \cdot b_{31}) \\
& + (a_{11} \cdot a_{24} - a_{14} \cdot a_{21}) \cdot (b_{11} \cdot b_{42} - b_{12} \cdot b_{41}) \\
& + (a_{12} \cdot a_{23} - a_{13} \cdot a_{22}) \cdot (b_{21} \cdot b_{32} - b_{22} \cdot b_{31}) \\
& + (a_{12} \cdot a_{24} - a_{14} \cdot a_{22}) \cdot (b_{21} \cdot b_{42} - b_{22} \cdot b_{41}) \\
& + (a_{13} \cdot a_{24} - a_{14} \cdot a_{23}) \cdot (b_{31} \cdot b_{42} - b_{32} \cdot b_{41})
\end{aligned}$$

$$a_{11} = e^{-(y-x)} \left[R_{11} e^{-x} + R_{21} e^{x} + R_{31} \cdot e^{-s_1 x} + R_{41} e^{s_1 x} \right]$$

$$a_{12} = \left[R_{12} e^{-x} + R_{22} e^{x} + R_{32} e^{-s_1 x} + R_{42} e^{s_1 x} \right]$$

$$a_{13} = e^{-s_1 (y-x)} \left[R_{13} e^{-x} + R_{23} e^{x} + R_{33} e^{-s_1 x} + R_{43} e^{s_1 x} \right]$$

$$a_{14} = \left[R_{14} e^{-x} + R_{24} e^{x} + R_{34} e^{-s_1 x} + R_{44} e^{s_1 x} \right]$$

$$a_{21} = e^{-(y-x)}\left[R_{11}e^{-x} - R_{21}e^{x} + s_1R_{31}e^{-s_1x} - s_1R_{41}e^{s_1x}\right]$$

$$a_{22} = \left[R_{12}e^{-x} - R_{22}e^{x} + s_1R_{32}e^{-s_1x} - s_1R_{42}e^{s_1x}\right]$$

$$a_{23} = e^{-s_2(y-x)}\left[R_{13}e^{-x} - R_{23}e^{x} + s_1R_{33}e^{-s_1x} - s_1R_{43}e^{s_1x}\right]$$

$$a_{24} = \left[R_{14}e^{-x} - R_{24}e^{x} + s_1R_{34}e^{-s_1x} - s_1R_{44}e^{s_1x}\right]$$

$$a_{11}\cdot a_{22} - a_{12}\cdot a_{21} = e^{-(y-x)}\cdot e^{s_1x}\cdot e^{x}\cdot$$

$$\begin{aligned}
&R_{11}\cdot\left[-2R_{22}e^{-x}e^{-s_1x} + (s_1-1)R_{32}e^{-2x}e^{-2s_1x} - (s_1+1)R_{42}e^{-2x}\right]\\
&+R_{21}\left[2R_{12}e^{-x}\cdot e^{-s_1x} + (1+s_1)R_{32}e^{-2s_1x} + (1-s_1)R_{42}\right]\\
&+R_{31}\left[(1-s_1)R_{12}e^{-2x}e^{-2s_1x} - (1+s_1)R_{22}e^{-2s_1x} - 2s_1R_{42}e^{-x}\cdot e^{-s_1x}\right]\\
&+R_{41}\left[(1+s_1)R_{12}e^{-2x} + (s_1-1)R_{22} + 2s_1R_{32}e^{-x}\cdot e^{-s_1x}\right]
\end{aligned}$$

$$a_{11}\cdot a_{23} - a_{13}\cdot a_{21} = e^{-(y-x)}\cdot e^{-s_2(y-x)}\cdot e^{s_1x}\cdot e^{x}\cdot$$

$$\begin{aligned}
&R_{11}\left[-2R_{23}e^{-x}e^{-s_1x} + (s_1-1)R_{33}e^{-2x}e^{-2s_1x} - (s_1+1)R_{43}e^{-2x}\right]\\
&+R_{21}\left[2R_{13}e^{-x}e^{-s_1x} + (1+s_1)R_{33}e^{-2s_1x} + (1-s_1)R_{43}\right]\\
&+R_{31}\left[(1-s_1)R_{13}e^{-2x}\cdot e^{-2s_1x} - (s_1+1)R_{23}e^{-2s_1x} - 2s_1R_{43}e^{-x}e^{-s_1x}\right]\\
&+R_{41}\left[(1+s_1)R_{13}e^{-2x} + (s_1-1)R_{23} + 2s_1R_{33}e^{-x}e^{-s_1x}\right]
\end{aligned}$$

$$a_{11}\cdot a_{24} - a_{14}\cdot a_{21} = e^{-(y-x)}\cdot e^{s_1x}\cdot e^{x}\cdot$$

$$\begin{aligned}
&R_{11}\left[-2R_{24}e^{-x}e^{-s_1x} + (s_1-1)R_{34}e^{-2x}e^{-2s_1x} - (1+s_1)R_{44}e^{-2x}\right]\\
&+R_{21}\left[2R_{14}e^{-x}e^{-s_1x} + (1+s_1)R_{34}e^{-2s_1x} + (1-s_1)R_{44}\right]\\
&+R_{31}\left[(1-s_1)R_{14}e^{-2x}e^{-2s_1x} - (1+s_1)R_{24}e^{-2s_1x} - 2s_1R_{44}e^{-x}\cdot e^{-s_1x}\right]\\
&+R_{41}\left[(1+s_1)R_{14}e^{-2x} + (s_1-1)R_{24} + 2s_1R_{34}e^{-x}\cdot e^{-s_1x}\right]
\end{aligned}$$

$$a_{12}.a_{23} - a_{13}.a_{22} = e^{-s_2(y-x)}.e^{s_1x}.e^{x}.$$

$$R_{12}\left[-2R_{23}e^{-x}e^{-s_1x} + (s_1-1)R_{33}e^{-2x}e^{-2s_1x} - (1+s_1)R_{43}e^{-2x}\right]$$
$$+R_{22}\left[2R_{13}e^{-x}e^{-s_1x} + (1+s_1)R_{33}e^{-2s_1x} + (1-s_1)R_{43}\right]$$
$$+R_{32}\left[(1-s_1)R_{13}e^{-2x}.e^{-2s_1x} - (1+s_1)R_{23}e^{-2s_1x} - 2s_1R_{43}e^{-x}e^{-s_1x}\right]$$
$$+R_{42}\left[(1+s_1)R_{13}e^{-2x} + (s_1-1)R_{23} + 2s_1R_{33}e^{-x}e^{-s_1x}\right]$$

$$a_{12}.a_{24} - a_{14}.a_{22} = e^{s_1x}.e^{x}.$$

$$R_{12}\left[-2R_{24}e^{-x}e^{-s_1x} + (s_1-1)R_{34}e^{-2x}e^{-2s_1x} - (1+s_1)R_{44}e^{-2x}\right]$$
$$+R_{22}\left[2R_{14}e^{-x}e^{-s_1x} + (1+s_1)R_{34}e^{-2s_1x} + (1-s_1)R_{44}\right]$$
$$+R_{32}\left[(1-s_1)R_{14}e^{-2x}e^{-2s_1x} - (1+s_1)R_{24}e^{-2s_1x} - 2s_1R_{44}e^{-x}e^{-s_1x}\right]$$
$$+R_{42}\left[(1+s_1)R_{14}e^{-2x} + (s_1-1)R_{24} + 2s_1R_{34}e^{-x}e^{-s_1x}\right]$$

$$a_{13}.a_{24} - a_{14}.a_{23} = e^{-s_2(y-x)}.e^{s_1x}.e^{x}.$$

$$R_{13}\left[-2R_{24}e^{-x}e^{-s_1x} + (s_1-1)R_{34}e^{-2x}e^{-2s_1x} - (1+s_1)R_{44}e^{-2x}\right]$$
$$+R_{23}\left[2R_{14}e^{-x}e^{-s_1x} + (1+s_1)R_{34}e^{-2s_1x} + (1-s_1)R_{44}\right]$$
$$+R_{33}\left[(1-s_1)R_{14}e^{-2x}e^{-2s_1x} - (1+s_1)R_{24}e^{-2s_1x} - 2s_1R_{44}e^{-x}e^{-s_1x}\right]$$
$$+R_{43}\left[(1+s_1)R_{14}e^{-2x} + (s_1-1)R_{24} + 2s_1R_{34}e^{-x}.e^{-s_1x}\right]$$

$$a_{11}.a_{22} - a_{12}.a_{21} = e^{-(y-x)}.e^{s_1x}.e^{x}.\ A_{12}$$
$$a_{11}.a_{23} - a_{13}.a_{21} = e^{-(y-x)}.e^{-s_2(y-x)}.e^{s_1x}.e^{x}.\ A_{13}$$
$$a_{11}.a_{24} - a_{14}.a_{21} = e^{-(y-x)}.e^{s_1x}.e^{x}.\ A_{14}$$
$$a_{12}.a_{23} - a_{13}.a_{22} = e^{-s_2(y-x)}.e^{s_1x}.e^{x}.\ A_{23}$$
$$a_{12}.a_{24} - a_{14}.a_{22} = e^{s_1x}.e^{x}.\ A_{24}$$
$$a_{13}.a_{24} - a_{14}.a_{23} = e^{-s_2(y-x)}.e^{s_1x}.e^{x}.\ A_{34}$$

$$b_{11} = Q_{11}.P_{11}.e^{-(z-y)} + Q_{12}.P_{21}.e^{(z-y)} + Q_{13}.P_{31}.e^{-s_3(z-y)} + Q_{14}.P_{41}\,e^{s_3(z-y)}$$

$$b_{12} = Q_{11}.P_{12} + Q_{12}.P_{22} + Q_{13}.P_{32} + Q_{14}.P_{42}$$

$$b_{21} = \left[Q_{21}.P_{11} + Q_{22}.P_{21} + Q_{23}.P_{31} + Q_{24}.P_{41}\right] e^{(y-x)}$$

$$b_{22} = \left[Q_{21}.P_{12} + Q_{22}.P_{22} + Q_{23}.P_{32} + Q_{24}.P_{42}\right] e^{(y-x)}$$

$$b_{31} = Q_{31}.P_{11} + Q_{32}.P_{21} + Q_{33}.P_{31} + Q_{34}.P_{41}$$

$$b_{32} = Q_{31}.P_{12} + Q_{32}.P_{22} + Q_{33}.P_{32} + Q_{34}.P_{42}$$

$$b_{41} = \left[Q_{41}.P_{11} + Q_{42}.P_{21} + Q_{43}.P_{31} + Q_{44}.P_{41}\right] e^{s_2(y-x)}$$

$$b_{42} = \left[Q_{41}.P_{12} + Q_{42}.P_{22} + Q_{43}.P_{32} + Q_{44}.P_{42}\right] e^{s_2(y-x)}$$

$$b_{11}.b_{22} - b_{12}.b_{21} = e^{(y-x)}.e^{(z-y)}.e^{s_3(z-y)}.$$

$$\Big\{ \left[Q_{11}.Q_{22} - Q_{12}.Q_{21}\right]\left[P_{11}.P_{22} - P_{12}.P_{21}\right] e^{-(z-y)}.e^{-s_3(z-y)}$$

$$+ \left[Q_{11}.Q_{24} - Q_{14}.Q_{21}\right]\left[P_{11}.P_{42} - P_{12}.P_{41}\right] e^{-2(z-y)}$$

$$+ \left[Q_{12}.Q_{23} - Q_{13}.Q_{22}\right]\left[P_{21}.P_{32} - P_{22}.P_{31}\right] e^{-2s_3(z-y)}$$

$$+ \left[Q_{13}.Q_{21} - Q_{11}.Q_{23}\right]\left[P_{31}.P_{12} - P_{32}.P_{11}\right] e^{-2(z-y)}.e^{-2s_3(z-y)}$$

$$+ \left[Q_{13}.Q_{24} - Q_{14}.Q_{23}\right]\left[P_{31}.P_{42} - P_{32}.P_{41}\right] e^{-(z-y)}.e^{-s_3(z-y)}$$

$$+ \left[Q_{14}.Q_{22} - Q_{12}.Q_{24}\right]\left[P_{41}.P_{22} - P_{42}.P_{21}\right] \Big\}$$

We write

$$p_{12} = \left[P_{11}.P_{22} - P_{12}.P_{21}\right] e^{-(z-y)}.e^{-s_3(z-y)}$$

$$p_{14} = \left[P_{11}.P_{42} - P_{12}.P_{41}\right] e^{-2(z-y)}$$

$$p_{23} = \left[P_{21}.P_{32} - P_{22}.P_{31}\right] e^{-2s_3(z-y)}\,.$$

$$p_{31} = \left[P_{31}.P_{12} - P_{32}.P_{11}\right] e^{-2(z-y)}.e^{-2s_3(z-y)}$$

$$p_{34} = \left[P_{31}.P_{42} - P_{32}.P_{41}\right] e^{-(z-y)}.e^{-s_3(z-y)}$$

$$p_{42} = \left[P_{41}.P_{22} - P_{42}.P_{21}\right]$$

$$b_{11}.b_{32} - b_{12}.b_{31} = e^{(z-y)}.e^{s_3(z-y)}.$$

$$\{[\Phi_{11}.\Phi_{32} - \Phi_{12}.\Phi_{31}].p_{12} + [\Phi_{11}.\Phi_{34} - \Phi_{14}\Phi_{31}].p_{14}$$
$$+[\Phi_{12}.\Phi_{33} - \Phi_{13}\Phi_{32}].p_{23} + [\Phi_{13}.\Phi_{31} - \Phi_{11}\Phi_{33}].p_{31}$$
$$+[\Phi_{13}.\Phi_{34} - \Phi_{14}.\Phi_{33}].p_{34} + [\Phi_{14}.\Phi_{32} - \Phi_{12}.\Phi_{34}]p_{42}\}$$

$$b_{11}.b_{42} - b_{12}.b_{41} = e^{s_2(y-x)}.e^{(z-y)}.e^{s_3(z-y)}.$$

$$\{[\Phi_{11}\Phi_{42} - \Phi_{12}.\Phi_{41}].p_{12} + [\Phi_{11}.\Phi_{44} - \Phi_{14}.\Phi_{41}].p_{14}$$
$$+[\Phi_{12}.\Phi_{43} - \Phi_{13}\Phi_{42}].p_{23} + [\Phi_{13}\Phi_{41} - \Phi_{11}.\Phi_{43}].p_{31}$$
$$+[\Phi_{13}.\Phi_{44} - \Phi_{14}\Phi_{43}].p_{34} + [\Phi_{14}\Phi_{42} - \Phi_{12}.\Phi_{44}].p_{42}\}$$

$$b_{21}.b_{32} - b_{22}.b_{31} = e^{(y-x)}.e^{(z-y)}.e^{s_3(z-y)}.$$

$$\{[\Phi_{21}.\Phi_{32} - \Phi_{22}.\Phi_{31}].p_{12} + [\Phi_{21}.\Phi_{34} - \Phi_{24}.\Phi_{31}].p_{14}$$
$$+[\Phi_{22}.\Phi_{33} - \Phi_{23}.\Phi_{32}].p_{23} + [\Phi_{23}.\Phi_{31} - \Phi_{21}.\Phi_{33}].p_{31}$$
$$+[\Phi_{23}.\Phi_{34} - \Phi_{24}.\Phi_{33}].p_{34} + [\Phi_{24}.\Phi_{32} - \Phi_{22}.\Phi_{34}].p_{42}\}$$

$$b_{21}.b_{42} - b_{22}.b_{41} = e^{s_2(y-x)}.e^{(z-y)}.e^{s_3(z-y)}.e^{(y-x)}$$

$$\{[\Phi_{21}.\Phi_{42} - \Phi_{22}.\Phi_{41}].p_{12} + [\Phi_{21}.\Phi_{44} - \Phi_{24}.\Phi_{41}].p_{14}$$
$$+[\Phi_{22}\Phi_{43} - \Phi_{23}.\Phi_{42}].p_{23} + [\Phi_{23}.\Phi_{41} - \Phi_{21}.\Phi_{43}].p_{31}$$
$$+[\Phi_{23}.\Phi_{44} - \Phi_{24}.\Phi_{43}]p_{34} + [\Phi_{24}.\Phi_{42} - \Phi_{22}.\Phi_{44}].p_{42}\}.$$

$$b_{31}.b_{42} - b_{32}.b_{41} = e^{s_2(y-x)}.e^{(z-y)}.e^{s_3(z-y)}.$$

$$\{[\Phi_{31}.\Phi_{42} - \Phi_{32}.\Phi_{41}].p_{12} + [\Phi_{31}.\Phi_{44} - \Phi_{34}.\Phi_{41}].p_{14}$$
$$+[\Phi_{32}.\Phi_{43} - \Phi_{33}.\Phi_{42}].p_{23} + [\Phi_{33}\Phi_{41} - \Phi_{31}.\Phi_{43}].p_{31}$$
$$+[\Phi_{33}.\Phi_{44} - \Phi_{34}.\Phi_{43}].p_{34} + [\Phi_{34}.\Phi_{42} - \Phi_{32}.\Phi_{44}].p_{42}\}$$

$$b_{11} \cdot b_{22} - b_{12} \cdot b_{21} = e^{(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot B_{12}$$

$$b_{11} \cdot b_{32} - b_{12} \cdot b_{31} = e^{(z-y)} \cdot e^{s_3(z-y)} \cdot B_{13}$$

$$b_{11} \cdot b_{42} - b_{12} \cdot b_{41} = e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot B_{14}$$

$$b_{21} \cdot b_{32} - b_{22} \cdot b_{31} = e^{(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot B_{23}$$

$$b_{21} \cdot b_{42} - b_{22} \cdot b_{41} = e^{(y-x)} \cdot e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot B_{24}$$

$$b_{31} \cdot b_{42} - b_{32} \cdot b_{41} = e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot B_{34}$$

The value of the denominator is then finally

$$c_{11} \cdot c_{22} - c_{12} \cdot c_{21} = e^{x} \cdot e^{s_1 x} \cdot e^{(y-x)} \cdot e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot$$

$$\begin{aligned}
\Big[& A_{12} \cdot B_{12} \cdot e^{-(y-x)} \cdot e^{-s_2(y-x)} \\
& + A_{13} \cdot B_{13} \cdot e^{-2(y-x)} \cdot e^{-2s_2(y-x)} \\
& + A_{14} \cdot B_{14}\, e^{-2(y-x)} \\
& + A_{23} \cdot B_{23}\, e^{-2s_2(y-x)} \\
& + A_{24} \cdot B_{24} \\
& + A_{34} \cdot B_{34}\, e^{-(y-x)} \cdot e^{-s_2(y-x)} \Big]
\end{aligned}$$

The term $A_{24} \cdot B_{24}$ contains the constant

$$\left[(1-s_1) R_{22} \cdot R_{44} + (s_1 - 1) R_{42} \cdot R_{24}\right] \cdot$$

$$\left[Q_{24} \cdot Q_{42} - Q_{22} \cdot Q_{44}\right] \cdot \left[P_{41} \cdot P_{22} - P_{42} \cdot P_{21}\right]$$

We write

$$c_{11} \cdot c_{22} - c_{12} \cdot c_{21} = e^{x} \cdot e^{s_1 x} \cdot e^{(y-x)} \cdot e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot C_{12}$$

3.2. Determination of the parameters B_4 and C_4.

We write

$$a'_{21} = \left[R_{11} \cdot e^{-2x} \cdot e^{-s_1 x} - R_{21}\, e^{-s_1 x} + s_1\, R_{31}\, e^{-x} \cdot e^{-2s_1 x} - s_1 R_{41}\, e^{-x}\right]$$

$$a'_{22} = \left[R_{12} \cdot e^{-2x} \cdot e^{-s_1 x} - R_{22}\, e^{-s_1 x} + s_1\, R_{32}\, e^{-x}\, e^{-2s_1 x} - s_1 R_{42}\, e^{-x}\right]$$

$$a'_{23} = \left[R_{13} \cdot e^{-2x} \cdot e^{-s_1 x} - R_{23}\, e^{-s_1 x} + s_1\, R_{33}\, e^{-x}\, e^{-2s_1 x} - s_1 R_{43}\, e^{-x}\right]$$

$$a'_{24} = \left[R_{14}\, e^{-2x} \cdot e^{-s_1 x} - R_{24}\, e^{-s_1 x} + s_1\, R_{34}\, e^{-x}\, e^{-2s_1 x} - s_1 R_{44}\, e^{-x}\right]$$

$$b'_{11} = \left[Q_{11} \cdot P_{11} \cdot e^{-2(z-y)} \cdot e^{-s_3(z-y)} + Q_{12} \cdot P_{21} \cdot e^{-s_3(z-y)} + Q_{13} \cdot P_{31}\, e^{-(z-y)}\, e^{-2s_3(z-y)} + Q_{14} \cdot P_{41}\, e^{-(z-y)}\right]$$

$$b'_{12} = \left[Q_{11} \cdot P_{12} + Q_{12} \cdot P_{22} + Q_{13}\, P_{32} + Q_{14} \cdot P_{42}\right]$$

$$b'_{21} = \left[Q_{21} \cdot P_{11} + Q_{22} \cdot P_{21} + Q_{23}\, P_{31} + Q_{24} \cdot P_{41}\right]$$

$$b'_{22} = \left[Q_{21} \cdot P_{12} + Q_{22} \cdot P_{22} + Q_{23}\, P_{32} + Q_{24} \cdot P_{42}\right]$$

$$b'_{31} = \left[Q_{31} \cdot P_{11} + Q_{32} \cdot P_{21} + Q_{33} \cdot P_{31} + Q_{34} \cdot P_{41}\right]$$

$$b'_{32} = \left[Q_{31} \cdot P_{12} + Q_{32} \cdot P_{22} + Q_{33} \cdot P_{32} + Q_{34} \cdot P_{42}\right]$$

$$b'_{41} = \left[Q_{41} \cdot P_{11} + Q_{42} \cdot P_{21} + Q_{43} \cdot P_{31} + Q_{44} \cdot P_{41}\right]$$

$$b'_{42} = \left[Q_{41} \cdot P_{12} + Q_{42} \cdot P_{22} + Q_{43} \cdot P_{32} + Q_{44} \cdot P_{42}\right]$$

We have that

$$B_4 = 8 s_1 s_2 s_3 (1-n_1)(1-n_2)(1-n_3) \cdot \frac{C_{22}}{C_{11} \cdot C_{22} - C_{12} \cdot C_{21}}$$

$$D_4 = -8 s_1 s_2 s_3 (1-n_1)(1-n_2)(1-n_3) \cdot \frac{C_{21}}{C_{11} \cdot C_{22} - C_{12} \cdot C_{21}}$$

$$c_{22} = a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} + a_{24} \cdot b_{42}$$

$$= e^{x} \cdot e^{s_1 x} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot \Big[e^{-(y-x)} a'_{21} \cdot b'_{12} + e^{(y-x)} \cdot a'_{22} \cdot b'_{22} + e^{-s_2(y-x)} a'_{23} \cdot b'_{32} + e^{s_2(y-x)} \cdot a'_{24} \cdot b'_{42} \Big]$$

$$= e^{x} \cdot e^{s_1 x} \cdot e^{(y-x)} \cdot e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot$$

$$\Big[e^{-2(y-x)} \cdot e^{-s_2(y-x)} \cdot a'_{21} \cdot b'_{12} + e^{-s_2(y-x)} a'_{22} \cdot b'_{22} + e^{-(y-x)} \cdot e^{-2s_2(y-x)} a'_{23} \cdot b'_{32} + e^{-(y-x)} \cdot a'_{24} \cdot b'_{42} \Big]$$

$$c_{21} = a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} b_{31} + a_{24} \cdot b_{41}$$

$$= e^{x} \cdot e^{s_1 x} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot \Big[e^{-(y-x)} a'_{21} \cdot b'_{11} + e^{(y-x)} \cdot a'_{22} \cdot b'_{21} + e^{-s_2(y-x)} a'_{23} \cdot b'_{31} + e^{s_2(y-x)} \cdot a'_{24} \cdot b'_{41} \Big]$$

$$= e^{x} \cdot e^{s_1 x} \cdot e^{(y-x)} \cdot e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot$$

$$\Big[e^{-2(y-x)} \cdot e^{-s_2(y-x)} \cdot a'_{21} \cdot b'_{11} + e^{-s_2(y-x)} a'_{22} \cdot b'_{21} + e^{-(y-x)} \cdot e^{-2s_2(y-x)} a'_{23} \cdot b'_{31} + e^{-(y-x)} \cdot a'_{24} \cdot b'_{41} \Big]$$

$$c_{22} = e^{x} \cdot e^{s_1 x} \cdot e^{(y-x)} \cdot e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot c'_{22}$$

$$c_{21} = e^{x} \cdot e^{s_1 x} \cdot e^{(y-x)} \cdot e^{s_2(y-x)} \cdot e^{(z-y)} \cdot e^{s_3(z-y)} \cdot c'_{21}$$

$$B_4 = 8 s_1 s_2 s_3 (1-n_1)(1-n_2)(1-n_3) \cdot \frac{c'_{22}}{C_{12}}$$

$$D_4 = -8 s_1 s_2 s_3 (1-n_1)(1-n_2)(1-n_3) \cdot \frac{c'_{21}}{C_{12}}$$

The numerators contain only negative exponents.

The denominator contains a constant and negative exponents.

4. Values of the parameters A_i, D_i.

4.1. Values of the parameters A_3, B_3, C_3, D_3.

The values of the parameters are obtained from the matrix equation in § 2.1.

$$\begin{pmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{pmatrix} = \frac{1}{2s_3(1-n_3)} \begin{pmatrix} P_{11} & P_{12} \\ P_{21}\,e^{(z-y)} & P_{22}\,e^{(z-y)} \\ P_{31} & P_{32} \\ P_{41}\,e^{(z-y).s_3} & P_{42}\,e^{s_3(z-y)} \end{pmatrix} \cdot \begin{pmatrix} B_4 \\ D_4 \end{pmatrix}$$

One obtains immediately the values of the parameters A_3 and C_3

$$A_3 = \frac{1}{2s_3(1-n_3)} \left[P_{11} . B_4 + P_{12} . D_4 \right]$$

$$C_3 = \frac{1}{2s_3(1-n_3)} \left[P_{31} . B_4 + P_{32} . D_4 \right]$$

The determination of the values of the parameters B_3 and D_3 need some more computation to insure convergency.

$$B_3 = \frac{e^{(z-y)}}{2s_3(1-n_3)} . \left[P_{21} . B_4 + P_{22} . D_4 \right]$$

$$D_3 = \frac{e^{s_3(z-y)}}{2s_3(1-n_3)} \left[P_{41} . B_4 + P_{42} . D_4 \right]$$

Those relations contain positive exponents which much disappear to avoid overflow problems.

$$B_3 = \frac{e^{(z-y)} .\ 8 s_1 s_2 s_3 (1-n_1)(1-n_2)(1-n_3)}{2s_3(1-n_3)\ .\ C_{12}} \left[P_{21} . B'_4 - P_{22} . D'_4 \right]$$

$$D_3 = \frac{e^{s_3(z-y)} .\ 8 s_1 s_2 s_3 (1-n_1)(1-n_2)(1-n_3)}{2s_3(1-n_3)\ .\ C_{12}} \left[P_{41} . B'_4 - P_{42} . D'_4 \right]$$

$$P_{21}.B'_4 - P_{22}.D'_4 =$$

$$P_{21}\Big[e^{-2(y-x)}e^{-s_2(y-x)}a'_{21}.b'_{12} + e^{-s_2(y-x)}a'_{22}.b'_{22} + e^{-(y-x)}e^{-2s_2(y-x)}a'_{23}.b'_{32} + e^{-(y-x)}a'_{24}.b'_{42}\Big]$$

$$-P_{22}\Big[e^{-2(y-x)}e^{-s_2(y-x)}a'_{21}.b'_{11} + e^{-s_2(y-x)}.a'_{22}.b'_{21} + e^{-(y-x)}e^{-2s_2(y-x)}a'_{23}.b'_{31} + e^{-(y-x)}a'_{24}.b'_{41}\Big] =$$

$$= e^{-2(y-x)}e^{-s_2(y-x)}a'_{21}\left[P_{21}.b'_{12} - P_{22}.b'_{11}\right]$$

$$+ e^{-s_2(y-x)}a'_{22}\left[P_{21}.b'_{22} - P_{22}b'_{21}\right]$$

$$+ e^{-(y-x)}.e^{-2s_2(y-x)}a'_{23}\left[P_{21}.b'_{32} - P_{22}.b'_{31}\right]$$

$$+ e^{-(y-x)}a'_{24}\left[P_{21}.b'_{42} - P_{22}.b'_{41}\right]$$

$$P_{21}.b'_{12} - P_{22}.b'_{11} =$$

$$P_{21}\Big[Q_{11}P_{12}e^{-2(z-y)}e^{-s_3(z-y)} + Q_{12}.P_{22}e^{-s_3(z-y)} + Q_{13}.P_{32}e^{-(z-y)}e^{-2s_3(z-y)} + Q_{14}.P_{42}e^{-(z-y)}\Big]$$

$$-P_{22}\Big[Q_{11}.P_{11} + Q_{12}.P_{21} + Q_{13}.P_{31} + Q_{14}P_{41}\Big] =$$

$$= e^{-(z-y)}P_{21}\Big[Q_{11}.P_{12}e^{-(z-y)}.e^{-s_3(z-y)} + Q_{13}.P_{32}e^{-2s_3(z-y)} + Q_{14}.P_{42}\Big]$$

$$-e^{-(z-y)}P_{22}\Big[Q_{11}.P_{11} + Q_{13}.P_{31} + Q_{14}.P_{41}\Big]$$

$$= e^{-(z-y)}.PB21$$

$$P_{21}.b'_{22} - P_{22}b'_{21} =$$

$$e^{-(z-y)}.P_{21}\Big[Q_{21}.P_{12}e^{-(z-y)}e^{-s_3(z-y)} + Q_{23}.P_{32}e^{-2s_3(z-y)} + Q_{24}.P_{42}\Big]$$

$$-e^{-(z-y)}.P_{22}\Big[Q_{21}.P_{11} + Q_{23}P_{31} + Q_{14}.P_{41}\Big]$$

$$= e^{-(z-y)}.PB22$$

$$P_{21}.b'_{32} - P_{22}.b'_{31} =$$

$$e^{-(z-y)}.P_{21}\left[\phi_{31}.P_{12}e^{-(z-y)}e^{-s_3(z-y)} + \phi_{33}.P_{32}e^{-2s_3(z-y)} + \phi_{34}.P_{42}\right]$$

$$- e^{-(z-y)}.P_{22}\left[\phi_{31}.P_{11} + \phi_{33}.P_{31} + \phi_{34}P_{41}\right]$$

$$= e^{-(z-y)}.PB_{23}$$

$$P_{21}.b'_{42} - P_{22}.b'_{41} =$$

$$e^{-(z-y)}.P_{21}\left[\phi_{41}.P_{12}e^{-(z-y)}e^{-s_3(z-y)} + \phi_{43}.P_{32}e^{-2s_3(z-y)} + \phi_{44}.P_{42}\right]$$

$$- e^{-(z-y)}.P_{22}\left[\phi_{41}.P_{11} + \phi_{43}.P_{31} + \phi_{44}.P_{41}\right]$$

$$= e^{-(z-y)}\,PB_{24}$$

The positive exponent $e^{(z-y)}$ can now be eliminated

$$B_3 = 4s_1s_2(1-n_1)(1-n_2).$$

$$\left[e^{-2(y-x)}.e^{-s_2(y-x)}.a'_{21}.PB_{21} + e^{-s_2(y-x)}a'_{22}.PB_{22} + e^{-(y-x)}.e^{-2s_2(y-x)}.a'_{23}.PB_{23} + e^{-(y-x)}a'_{24}.PB_{24}\right].\frac{1}{C_{12}}$$

and the numerator contains again only negative exponents.

$$P_{41}.B'_4 - P_{42}.D'_4 =$$

$$P_{41}\left[e^{-2(y-x)}.e^{-s_2(y-x)}a'_{21}.b'_{12} + e^{-s_2(y-x)}a'_{22}.b'_{22} + e^{-(y-x)}e^{-2s_2(y-x)}a'_{23}.b'_{32} + e^{-(y-x)}a'_{24}.b'_{42}\right]$$

$$- P_{42}\left[e^{-2(y-x)}.e^{-s_2(y-x)}a'_{21}.b'_{11} + e^{-s_2(y-x)}a'_{22}.b'_{21} + e^{-(y-x)}e^{-2s_2(y-x)}a'_{23}.b'_{31} + e^{-(y-x)}a'_{24}.b'_{41}\right] =$$

$$= e^{-2(y-x)}e^{-s_2(y-x)}a'_{21}\left[P_{41}.b'_{12} - P_{42}.b'_{11}\right] + e^{-s_2(y-x)}a'_{22}\left[P_{41}.b'_{22} - P_{42}.b'_{21}\right]$$

$$+ e^{-(y-x)}e^{-2s_2(y-x)}a'_{23}\left[P_{41}.b'_{32} - P_{42}.b'_{31}\right] + e^{-(y-x)}a'_{24}\left[P_{41}.b'_{42} - P_{42}.b'_{41}\right]$$

$$P_{41}.b'_{12} - P_{42}.b'_{11} =$$

$$P_{41}\left[Q_{11}.P_{12}e^{-2(z-y)}e^{-s_3(z-y)} + Q_{12}P_{22}e^{-s_3(z-y)} + Q_{13}P_{32}e^{-(z-y)}e^{-2s_3(z-y)} + Q_{14}.P_{42}e^{-(z-y)}\right]$$
$$- P_{42}\left[Q_{11}.P_{11} + Q_{12}.P_{21} + Q_{13}P_{31} + Q_{14}P_{41}\right]$$

$$= e^{-s_3(z-y)}.P_{41}\left[Q_{11}.P_{12}e^{-2(z-y)} + Q_{12}.P_{22} + Q_{13}.P_{32}e^{-(z-y)}e^{-s_3(z-y)}\right]$$
$$- e^{-s_3(z-y)}.P_{42}\left[Q_{11}P_{11} + Q_{12}.P_{21} + Q_{13}P_{31}\right]$$

$$= e^{-s_3(z-y)}\,PB_{41}$$

$$P_{41}.b'_{22} - P_{42}.b'_{21} =$$
$$= e^{-s_3(z-y)}.P_{41}\left[Q_{21}.P_{12}e^{-2(z-y)} + Q_{22}.P_{22} + Q_{23}.P_{32}e^{-(z-y)}.e^{-s_3(z-y)}\right]$$
$$- e^{-s_3(z-y)}.P_{42}\left[Q_{21}.P_{11} + Q_{22}.P_{21} + Q_{23}.P_{31}\right]$$

$$= e^{-s_3(z-y)}\,PB_{42}$$

$$P_{41}.b'_{32} - P_{42}.b'_{31} =$$
$$= e^{-s_3(z-y)}.P_{41}\left[Q_{31}.P_{12}e^{-2(z-y)} + Q_{32}.P_{22} + Q_{33}.P_{32}.e^{-(z-y)}.e^{-s_3(z-y)}\right]$$
$$- e^{-s_3(z-y)}.P_{42}\left[Q_{31}.P_{11} + Q_{32}.P_{21} + Q_{33}.P_{31}\right]$$

$$= e^{-s_3(z-y)}\,PB_{43}$$

$$P_{41}.b'_{42} - P_{42}.b'_{41} =$$
$$= e^{-s_3(z-y)}P_{41}\left[Q_{41}.P_{12}e^{-2(z-y)} + Q_{42}.P_{22} + Q_{43}.P_{32}e^{-(z-y)}e^{-s_3(z-y)}\right]$$
$$- e^{-s_3(z-y)}P_{42}\left[Q_{41}.P_{11} + Q_{42}.P_{21} + Q_{43}.P_{31}\right]$$

$$= e^{-s_3(z-y)}\,PB_{44}$$

and the positive exponent $e^{s_3(z-y)}$ can be eliminated

$$D_3 = 4s_1s_2(1-n_1)(1-n_2).$$
$$\left[e^{-2(y-x)}e^{-s_2(y-x)}a'_{21}.PB_{41} + e^{-s_2(y-x)}a'_{22}.PB_{42}\right.$$
$$\left. + e^{-(y-x)}e^{-2s_2(y-x)}a'_{23}.PB_{43} + e^{-(y-x)}a'_{24}.PB_{44}\right].\frac{1}{C_{12}}.$$

4.2. Values of the parameters A_2, B_2, C_2, D_2.

The values of the parameters A_2 and C_2 are immediately obtained from the matrix equation in § 2.2

$$A_2 = \frac{1}{2s_2(1-n_2)} \left[Q_{11} \cdot A_3 \, e^{-(z-y)} + Q_{12} \cdot B_3 + Q_{13} \cdot C_3 \, e^{-s_3(z-y)} + Q_{14} \cdot D_3 \right]$$

$$C_2 = \frac{1}{2s_2(1-n_2)} \left[Q_{31} \cdot A_3 \, e^{-(z-y)} + Q_{32} \cdot B_3 + Q_{33} \cdot C_3 \, e^{-s_3(z-y)} + Q_{34} \cdot D_3 \right]$$

The values of the parameters B_2 and D_2 are obtained from next matrix equation (§ 3):

$$\begin{pmatrix} A_2 \\ B_2 \\ C_2 \\ D_2 \end{pmatrix} = \frac{1}{4s_2s_3(1-n_2)(1-n_3)} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{pmatrix} \cdot \begin{vmatrix} B_4 \\ D_4 \end{vmatrix}$$

The necessity of convergency needs again some more computation.

$$B_2 = \frac{1}{4s_2s_3(1-n_2)(1-n_3)} \cdot \left[b_{21} \cdot B_4 + b_{22} \cdot D_4 \right]$$

$$= \frac{2\,s_1(1-n_1)}{C_{12}} \left[c'_{22} \cdot b_{21} - c'_{21} \cdot b_{22} \right]$$

$$D_2 = \frac{1}{4s_2s_3(1-n_2)(1-n_3)} \left[b_{41} \cdot B_4 + b_{42} \, D_4 \right]$$

$$= \frac{2\,s_1(1-n_1)}{C_{12}} \left[c'_{22} \cdot b_{41} - c'_{21} \cdot b_{42} \right]$$

$$c'_{22}.b_{21} - c'_{21}.b_{22} =$$

$$\Big[e^{-2(y-x)}e^{-s_2(y-x)}a'_{21}.b'_{12} + e^{-s_2(y-x)}a'_{22}.b'_{22}$$
$$+ e^{-(y-x)}e^{-2s_2(y-x)}a'_{23}.b'_{32} + e^{-(y-x)}a'_{24}.b'_{42}\Big]b_{21}$$
$$- \Big[e^{-2(y-x)}.e^{-s_2(y-x)}a'_{21}.b'_{11} + e^{-s_2(y-x)}a'_{22}.b'_{21}$$
$$+ e^{-(y-x)}.e^{-2s_2(y-x)}a'_{23}.b'_{31} + e^{-(y-x)}a'_{24}b'_{41}\Big]b_{22}$$

$$= e^{-2(y-x)}.e^{-s_2(y-x)}a'_{21}\left[b'_{12}.b_{21} - b'_{11}.b_{22}\right]$$
$$+ e^{-s_2(y-x)}a'_{22}\left[b'_{22}.b_{21} - b'_{21}.b_{22}\right]$$
$$+ e^{-(y-x)}.e^{-2s_2(y-x)}a'_{23}\left[b'_{32}.b_{21} - b'_{31}.b_{22}\right]$$
$$+ e^{-(y-x)}a'_{24}\left[b'_{42}.b_{21} - b'_{41}.b_{22}\right]$$

$$= -e^{-2(y-x)}.e^{-s_2(y-x)}a'_{21}.e^{(y-x)}.B_{12}$$
$$+ e^{-(y-x)}.e^{-2s_2(y-x)}a'_{23}.e^{(y-x)}.B_{23}$$
$$+ e^{-(y-x)}.a'_{24}.e^{(y-x)}.B_{24}$$

$$B_2 = \frac{2s_1(1-h_1)}{C_{12}}\left[a'_{24}.B_{24} + e^{-2s_2(y-x)}a'_{23}.B_{23} - e^{-(y-x)}.e^{-s_2(y-x)}.a'_{21}.B_{12}\right]$$

The positive exponent, $e^{(y-x)}$ included in b_{21} and b_{22}, has disappeared.

$$c'_{22}.b_{41} - c'_{21}.b_{42} =$$

$$e^{-2(y-x)}.e^{-s_2(y-x)}a'_{21}\left[b'_{12}.b_{41} - b'_{11}.b_{42}\right]$$
$$+ e^{-s_2(y-x)}a'_{22}\left[b'_{22}.b_{41} - b'_{21}.b_{42}\right]$$
$$+ e^{-(y-x)}.e^{-2s_2(y-x)}a'_{23}\left[b'_{32}.b_{41} - b'_{31}.b_{42}\right]$$
$$+ e^{-(y-x)}a'_{24}\left[b'_{42}.b_{41} - b'_{41}.b_{42}\right]$$

$$c'_{22}.b_{41} - c'_{21}.b_{42} =$$

$$- e^{-2(y-x)}.e^{-s_2(y-x)} a'_{21}\, e^{s_2(y-x)}.\, B_{14}$$
$$- e^{-s_2(y-x)}\, a'_{22}\,.\, e^{s_2(y-x)}.\, B_{24}$$
$$- e^{-(y-x)}.\, e^{-2s_2(y-x)}.\, a'_{23}\,.\, e^{s_2(y-x)}.\, B_{34}$$

$$D_2 = -\frac{2s_1(1-n_1)}{C_{12}}\left[e^{2(y-x)} a'_{21}.B_{14} + a'_{22}.B_{24} + a'_{23}.e^{-(y-x)}.e^{-s_2(y-x)}.B_{34}\right]$$

The positive exponent, $e^{s_2(y-x)}$ included in b_{41} and b_{42}, has disappeared.

The numerators of B_2 and D_2 contain only negative exponents because of the presence of the factors a'_{21}, a'_{22}, a'_{23} and a'_{24}.

4.3. Values of the parameters A_1, B_1, C_1, D_1.

The values of the parameters A_1 and C_1 are immediately obtained from the matrix equation in § 2.3.

$$A_1 = \frac{1}{2s_1(1-n_1)}\left[R_{11} e^{-(y-x)}.A_2 + R_{12}.B_2 + R_{13}\, e^{-s_2(y-x)}.C_2 + R_{14}.D_2\right]$$

$$C_1 = \frac{1}{2s_1(1-n_1)}\left[R_{31} e^{-(y-x)}.A_2 + R_{32}.B_2 + R_{33} e^{-s_2(y-x)}.C_2 + R_{34}.D_2\right]$$

The values of the parameters B_1 and D_1 are obtained from the boundary conditions at the surface

$$B_1 = \frac{1}{s_1 - 1}\left[s_1 - (1+s_1)\, A_1 e^{-x} - 2s_1 C_1\, e^{-s_1 x}\right]$$

$$D_1 = -\frac{1}{s_1 - 1}\left[1 - 2A_1 e^{-x} - (1+s_1) C_1\, e^{-s_1 x}\right]$$

2. Expression of the boundary conditions in matrixform.

2.1. At the third interface.

In the equations at the third interface, A_4, or C_4, is replaced by its value obtained from the fixed bottom condition.

We write the conditions at the third interface in matrixform

APPENDIX 4

ALGEBRAICAL ANALYSIS OF ANISOTROPIC LAYERED SYSTEMS WITH FIXED BOTTOM AND FULL SLIP CONDITIONS AT THE TWO FIRST INTERFACES

SYMBOLS

We write

$$A_i m^2 n_i (1+\mu_i) e^{m(H_1+H_2+\dots H_i)} = A_i$$

$$B_i m^2 n_i (1+\mu_i) e^{-m(H_1+H_2+\dots H_{i-1})} = B_i$$

$$C_i m^2 n_i s_i (n_i+\mu_i) e^{m s_i (H_1+H_2+\dots H_i)} = C_i$$

$$D_i m^2 n_i s_i (n_i+\mu_i) e^{-m s_i (H_1+H_2+\dots H_{i-1})} = D_i$$

$$F_1 = \frac{E_1}{E_2} \qquad k_1 = \frac{E_2}{E_3} \qquad L_1 = \frac{E_3}{E_4}$$

$$s_i = \left(\frac{n_i - \mu_i^2}{n_i^2 - \mu_i^2}\right)^{1/2}$$

$$x = mH_1$$

$$y = m(H_1 + H_2)$$

$$z = m(H_1 + H_2 + H_3)$$

$$t = m(H_1 + H_2 + H_3 + H_4)$$

where H_1, H_2, H_3, H_4 are the thicknesses of the successive layers.
The index 1 applies to the surface layer.
We shall successively analyse a two layered, a three layered and a four layered system.

Chapter 1. The two layered system.

1. The boundary conditions.

Boundary conditions at the surface (z = 0):

$$\sigma_z = p \qquad A_1 e^{-x} + B_1 + C_1 e^{-s_1 x} + D_1 = 1$$

$$\tau_{rz} = 0 \qquad A_1 e^{-x} - B_1 + s_1 C_1 e^{-s_1 x} - s_1 D_1 = 0$$

Boundary conditions at the interface ($z = H_1$):

$$\sigma_z : \; A_1 + B_1 e^{-x} + C_1 + D_1 e^{-s_1 x} = A_2 e^{-(y-x)} + B_2 + C_2 e^{-s_2(y-x)} + D_2$$

$$\tau_{rz} : \; A_1 - B_1 e^{-x} + s_1 C_1 - s_1 D_1 e^{-s_1 x} = 0$$

$$A_2 e^{-(y-x)} - B_2 + s_2 C_2 e^{-s_2(y-x)} - s_2 D_2 = 0$$

$$w : \; (1+\mu_1) A_1 - (1+\mu_1) B_1 e^{-x} + s_1 (n_1+\mu_1) C_1 - s_1 (n_1+\mu_1) D_1 e^{-s_1 x} =$$

$$F_1 \left[(1+\mu_2) A_2 e^{-(y-x)} - (1+\mu_2) B_2 + s_2 (n_2+\mu_2) C_2 e^{-s_2(y-x)} - s_2 (n_2+\mu_2) D_2 \right]$$

Boundary conditions at the bottom ($z = H_1 + H_2$):

$$w = 0 \qquad (1+\mu_2) A_2 - (1+\mu_2) B_2 e^{-(y-x)} + s_2 (n_2+\mu_2) C_2 - s_2 (n_2+\mu_2) D_2 e^{-s_2(y-x)} = 0$$

If $s_2 > 1$ $\qquad C_2 = 0$

$$A_2 e^{-(y-x)} = B_2 e^{-2(y-x)} + \frac{s_2 (n_2+\mu_2)}{(1+\mu_2)} D_2 e^{-(y-x)} e^{-s_2(y-x)}$$

If $s_2 < 1$ $\qquad A_2 = 0$

$$C_2 e^{-s_2(y-x)} = \frac{(1+\mu_2)}{s_2 (n_2+\mu_2)} B_2 e^{-(y-x)} e^{-s_2(y-x)} + D_2 e^{-2 s_2 (y-x)}$$

2. Resolution of the system of 6 boundary equations.

We add and substract the surface conditions

$$2 A_1 e^{-x} = 1 - (1+s_1) C_1 e^{-s_1 x} - (1-s_1) D_1$$

$$2 B_1 = 1 - (1-s_1) C_1 e^{-s_1 x} - (1+s_1) D_1$$

We add and substract the first two conditions at the first interface

$$2A_1 + (1+s_1)C_1 + (1-s_1)D_1 e^{-s_1 x} = [A2]$$

$$2B_1 e^{-x} + (1-s_1)C_1 + (1+s_1)D_1 e^{-s_1 x} = [A2]$$

where, taken in account the bottom condition,

if $s_2 > 1$

$$[A2] = \left[1 + e^{-2(y-x)}\right] B_2 + \left[1 + \frac{s_2(n_2+\mu_2)}{(1+\mu_2)} e^{-(y-x)} \cdot e^{-s_2(y-x)}\right] D_2$$

if $s_2 < 1$

$$[A2] = \left[1 + \frac{1+\mu_2}{s_2(n_2+\mu_2)} e^{-(y-x)} e^{-s_2(y-x)}\right] B_2 + \left[1 + e^{-2s_2(y-x)}\right] D_2$$

We replace A_1 and B_1 by their values in function of C_1 and D_1

$$(1+s_1)C_1\left[1 - e^{x} \cdot e^{-s_1 x}\right] + (1-s_1)D_1\left[e^{-s_1 x} - e^{x}\right] = [A2] - e^{x}$$

$$(1-s_1)C_1\left[1 - e^{-x} e^{-s_1 x}\right] + (1+s_1)D_1\left[e^{-s_1 x} - e^{-x}\right] = [A2] - e^{-x}$$

We solve the system

$$C_1 = \Big\{ [A2]\left[2s_1 e^{-x} e^{-s_1 x} - (1+s_1)e^{-2x} + (1-s_1)\right] + \left[2s_1 e^{-x} - (1+s_1)e^{-s_1 x} + (1-s_1)e^{-2x} e^{-s_1 x}\right] \Big\} \frac{1}{\nabla_1}$$

$$D_1 = \Big\{ [A2]\left[2s_1 e^{-x} - (1+s_1)e^{-s_1 x} + (1-s_1)e^{-2x} e^{-s_1 x}\right] + \left[2s_1 e^{-x} e^{-s_1 x} - (1+s_1)e^{-2x} + (1-s_1)\right] \Big\} \frac{1}{\nabla_1}$$

$$\nabla_1 = 8s_1 e^{-x} e^{-s_1 x} - (1+s_1)^2\left[e^{-2x} + e^{-2s_1 x}\right] + (1-s_1)^2\left[1 + e^{-2x} e^{-2s_1 x}\right]$$

For $m = \infty$ $\quad \nabla_1 = (1-s_1)^2$

We transform the w-condition utilizing the τ_{rz}-conditions

$$s_1(n_1-1)\,C_1 - s_1(n_1-1)\,D_1 e^{-s_1 x} = F_1\left[s_2(n_2-1)\,C_2 e^{-s_2(y-x)} - s_2(n_2-1)\,D_2\right]$$

and with

$$F_2 = F_1\,\frac{s_2(n_2-1)}{s_1(n_1-1)}$$

$$C_1 - D_1 e^{-s_1 x} = F_2\left[C_2 e^{-s_2(y-x)} - D_2\right]$$

We replace C_1 and D_1 by their values

$$C_1 - D_1 e^{-s_1 x} =$$

$$\Big\{[A2]\left[(1-s_1) - (1+s_1)e^{-2x} + (1+s_1)e^{-2s_1 x} - (1-s_1)e^{-2x}e^{-2s_1 x}\right]$$
$$+\left[2s_1 e^{-x}\left(1-e^{-2s_1 x}\right) + 2e^{-2x}e^{-s_1 x} - 2e^{-s_1 x}\right]\Big\}\frac{1}{\nabla_1}$$

$$= \Big\{[A2]\cdot R_1 + R_2\Big\}\frac{1}{\nabla_1}$$

$$= F_2\left[C_2 e^{-s_2(y-x)} - D_2\right]$$

with

$$R_3 = \frac{F_2\,\nabla_1}{R_1}$$

$$[A2] - R_3\left[C_2 e^{-s_2(y-x)} - D_2\right] = -\frac{R_2}{R_1}$$

This relation becomes

If $s_2 > 1$

$$\left[1+e^{-2(y-x)}\right]B_2 + \left[1 + \frac{s_2(n_2+\mu_2)}{(1+\mu_2)}\,e^{-(y-x)}e^{-s_2(y-x)} + R_3\right]D_2 = -\frac{R_2}{R_1}$$

If $s_2 < 1$

$$\left[1 + (1-R_3)\,\frac{1+\mu_2}{s_2(n_2+\mu_2)}\,e^{-(y-x)}e^{-s_2(y-x)}\right]B_2$$
$$+\left[1 + (1-R_3)\,e^{-2s_2(y-x)} + R_3\right]D_2 = -\frac{R_2}{R_1}$$

Together with the τ_{rz}-condition

If $s_2 > 1$

$$\left[e^{-2(y-x)} - 1\right] B_2 + \left[\frac{s_2(n_2+\mu_2)}{(1+\mu_2)} e^{-(y-x)} e^{-s_2(y-x)} - s_2\right] D_2 = 0$$

If $s_2 < 1$

$$\left[\frac{(1+\mu_2)}{(n_2+\mu_2)} e^{-(y-x)} e^{-s_2(y-x)} - 1\right] B_2 + \left[s_2 e^{-2s_2(y-x)} - s_2\right] D_2 = 0$$

we solve the system.

If $s_2 > 1$

$$B_2 = -\frac{R_2}{R_1} \frac{\frac{s_2(n_2+\mu_2)}{(1+\mu_2)} e^{-(y-x)} e^{-s_2(y-x)} - s_2}{\nabla_2}$$

$$D_2 = \frac{R_2}{R_1} \frac{e^{-2(y-x)} - 1}{\nabla_2}$$

$$\nabla_2 = (1-s_2+R_3) - (1+s_2+R_3) e^{-2(y-x)} + \frac{2s_2(n_2+\mu_2)}{(1+\mu_2)} e^{-(y-x)} e^{-s_2(y-x)}$$

If $s_2 < 1$

$$B_2 = -\frac{R_2}{R_1} \frac{s_2 e^{-2s_2(y-x)} - s_2}{\nabla_2}$$

$$D_2 = \frac{R_2}{R_1} \frac{\frac{1+\mu_2}{n_2+\mu_2} e^{-(y-x)} e^{-s_2(y-x)} - 1}{\nabla_2}$$

$$\nabla_2 = (1-s_2+R_3) + (1+s_2-R_3) e^{-2s_2(y-x)} - \frac{2(1+\mu_2)}{(n_2+\mu_2)} e^{-(y-x)} e^{-s_2(y-x)}$$

3. Relations for the parameters.

$$C_1 = \Big\{ [A2]\big[2s_1 e^{-x} e^{-s_1 x} - (1+s_1) e^{-2x} + (1-s_1)\big]$$
$$+ \big[2s_1 e^{-x} - (1+s_1) e^{-s_1 x} + (1-s_1) e^{-2x} e^{-s_1 x}\big]\Big\} \frac{1}{\nabla_1}$$

$$D_1 e^{-s_1 x} = \Big\{ [A2]\big[2s_1 e^{-x} e^{-s_1 x} - (1+s_1) e^{-2s_1 x} + (1-s_1) e^{-2x} e^{-2s_1 x}\big]$$
$$+ \big[2s_1 e^{-x} e^{-2s_1 x} - (1+s_1) e^{-2x} e^{-s_1 x} + (1-s_1) e^{-s_1 x}\big]\Big\} \frac{1}{\nabla_1}$$

$$A_1 = \frac{1}{2} \Big\{ [A2] - (1+s_1) C_1 - (1-s_1) D_1 e^{-s_1 x} \Big\}$$

The values of the parameters B_1 and D_1 are obtained from the surface conditions

$$B_1 = \frac{1}{s_1 - 1} \big[s_1 - (1+s_1) A_1 e^{-x} - 2s_1 C_1 e^{-s_1 x} \big]$$

$$D_1 = - \frac{1}{s_1 - 1} \big[1 - 2A_1 e^{-x} - (1+s_1) C_1 e^{-s_1 x} \big]$$

Chapter 3. The three layered structure.

1. Supplementary boundary conditions.

Boundary conditions at the second interface ($z = H_1 + H_2$):

$$\sigma_2: \quad A_2 + B_2 e^{-(y-x)} + C_2 + D_2 e^{-s_2(y-x)} = A_3 e^{-(z-y)} + B_3 + C_3 e^{-s_3(z-y)} + D_3$$

$$\tau_{r2}: \quad A_2 - B_2 e^{-(y-x)} + s_2 C_2 - s_2 D_2 e^{-s_2(y-x)} = 0$$

$$A_3 e^{-(z-y)} - B_3 + s_3 C_3 e^{-s_3(z-y)} - s_3 D_3 = 0$$

$$w: \quad (1+\mu_2) A_2 - (1+\mu_2) B_2 e^{-(y-x)} + s_2(n_2+\mu_2) C_2 - s_2(n_2+\mu_2) D_2 e^{-s_2(y-x)} =$$

$$K_1 \left[(1+\mu_3) A_3 e^{-(z-y)} - (1+\mu_3) B_3 + s_3(n_3+\mu_3) C_3 e^{-s_3(z-y)} - s_3(n_3+\mu_3) D_3 \right]$$

Boundary conditions at the bottom ($z = H_1 + H_2 + H_3$):

$$w=0 \quad (1+\mu_3) A_3 - (1+\mu_3) B_3 e^{-(z-y)} + s_3(n_3+\mu_3) C_3 - s_3(n_3+\mu_3) D_3 e^{-s_3(z-y)} = 0$$

If $\quad s_3 > 1 \qquad C_3 = 0$

$$A_3 e^{-(z-y)} = B_3 e^{-2(z-y)} + \frac{s_3(n_3+\mu_3)}{(1+\mu_3)} D_3 e^{-(z-y)} e^{-s_3(z-y)}$$

If $\quad s_3 < 1 \qquad A_3 = 0$

$$C_3 e^{-s_3(z-y)} = \frac{(1+\mu_3)}{s_3(n_3+\mu_3)} B_3 e^{-(z-y)} e^{-s_3(z-y)} + D_3 e^{-2s_3(z-y)}$$

2. Resolution of the system of 10 boundary equations.

We add and substract the surface conditions

$$2A_1 e^{-x} = 1 - (1+s_1) C_1 e^{-s_1 x} - (1-s_1) D_1$$

$$2B_1 = 1 - (1-s_1) C_1 e^{-s_1 x} - (1+s_1) D_1$$

We add and substract the first two conditions at the first interface

$$2A_1 + (1+s_1) C_1 + (1-s_1) D_1 e^{-s_1 x} = [A2]$$

$$2B_1 e^{-x} + (1-s_1) C_1 + (1+s_1) D_1 e^{-s_1 x} = [A2]$$

where

$$[A2] = A_2 e^{-(y-x)} + B_2 + C_2 e^{-s_2(y-x)} + D_2$$

We replace A_1 and B_1 by their values in function of C_1 and D_1

$$(1+s_1) C_1 \left[1 - e^{x} e^{-s_1 x}\right] + (1-s_1) D_1 \left[e^{-s_1 x} - e^{x}\right] = [A2] - e^{x}$$

$$(1-s_1) C_1 \left[1 - e^{-x} e^{-s_1 x}\right] + (1+s_1) D_1 \left[e^{-s_1 x} - e^{-x}\right] = [A2] - e^{-x}$$

We solve the system

$$C_1 = \Big\{ [A2]\left[2s_1 e^{-x} e^{-s_1 x} - (1+s_1) e^{-2x} + (1-s_1)\right]$$
$$+ \left[2s_1 e^{-x} - (1+s_1) e^{-s_1 x} + (1-s_1) e^{-2x} e^{-s_1 x}\right] \Big\} \frac{1}{\nabla_1}$$

$$D_1 = \Big\{ [A2]\left[2s_1 e^{-x} - (1+s_1) e^{-s_1 x} + (1-s_1) e^{-2x} e^{-s_1 x}\right]$$
$$+ \left[2s_1 e^{-x} e^{-s_1 x} - (1+s_1) e^{-2x} + (1-s_1)\right] \Big\} \frac{1}{\nabla_1}$$

$$\nabla_1 = 8 s_1 e^{-x} e^{-s_1 x} - (1+s_1)^2 \left[e^{-2x} + e^{-2s_1 x}\right] + (1-s_1)^2 \left[1 + e^{-2x} e^{-2s_1 x}\right]$$

We transform the w-condition utilizing the τ_{rz}-conditions

$$s_1(n_1-1) C_1 - s_1(n_1-1) D_1 e^{-s_1 x} = F_1 \left[s_2(n_2-1) C_2 e^{-s_2(y-x)} - s_2(n_2-1) D_2\right]$$

and with

$$F_2 = F_1 \frac{s_2(n_2-1)}{s_1(n_1-1)}$$

$$C_1 - D_1 e^{-s_1 x} = F_2 \left[C_2 e^{-s_2(y-x)} - D_2\right]$$

We replace C_1 and D_1 by their values

$$C_1 - D_1 e^{-s_1 x} =$$
$$\Big\{ [A2]\left[(1-s_1) - (1+s_1) e^{-2x} + (1+s_1) e^{-2s_1 x} - (1-s_1) e^{-2x} e^{-2s_1 x}\right]$$
$$+ \left[2s_1 e^{-x}(1 - e^{-2s_1 x}) + 2 e^{-2x} e^{-s_1 x} - 2 e^{-s_1 x}\right] \Big\} \cdot \frac{1}{\nabla_1}$$
$$= \Big\{ [A2] \cdot R_1 + R_2 \Big\} \cdot \frac{1}{\nabla_1}$$

$$\left\{[A2]\,R_1 + R_2\right\}\frac{1}{\nabla_1} = F_2\left[C_2 e^{-s_2(y-x)} - D_2\right]$$

$$[A2] - \frac{F_2\nabla_1}{R_1}\left[C_2 e^{-s_2(y-x)} - D_2\right] = -\frac{R_2}{R_1}$$

Writing $R_3 = \frac{F_2\nabla_1}{R_1}$, we obtain the system

$$A_2 e^{-(y-x)} + B_2 + (1-R_3)\,C_2 e^{-s_2(y-x)} + (1+R_3)\,D_2 = -\frac{R_2}{R_1}$$

$$A_2 e^{-(y-x)} - B_2 + s_2 C_2 e^{-s_2(y-x)} - s_2 D_2 = 0$$

and by adding and substracting

$$2A_2 e^{-(y-x)} = -(1+s_2-R_3)\,C_2 e^{-s_2(y-x)} - (1-s_2+R_3)\,D_2 - \frac{R_2}{R_1}$$

$$2B_2 = -(1-s_2-R_3)\,C_2 e^{-s_2(y-x)} - (1+s_2+R_3)\,D_2 - \frac{R_2}{R_1}$$

We add and substract the first two conditions at the second interface

$$2A_2 + (1+s_2)\,C_2 + (1-s_2)\,D_2 e^{-s_2(y-x)} = [A3]$$

$$2B_2 e^{-(y-x)} + (1-s_2)\,C_2 + (1+s_2)\,D_2 e^{-s_2(y-x)} = [A3]$$

where, taken in account the bottom conditions,

if $s_3 > 1$

$$[A3] = \left[1 + e^{-2(z-y)}\right] B_3 + \left[1 + \frac{s_3(\mu_3+\mu_3)}{(1+\mu_3)}\, e^{-(z-y)}\, e^{-s_3(z-y)}\right] D_3$$

if $s_3 < 1$

$$[A3] = \left[1 + \frac{(1+\mu_3)}{s_3(\mu_3+\mu_3)}\, e^{-(z-y)}\, e^{-s_3(z-y)}\right] B_3 + \left[1 + e^{-2s_3(z-y)}\right] D_3$$

We replace A_2 and B_2 by their values from the first interface conditions

$$C_2\left[(1+s_2) - (1+s_2-R_3)\,e^{(y-x)}\,e^{-s_2(y-x)}\right]$$
$$+ D_2\left[(1-s_2)\,e^{-s_2(y-x)} - (1-s_2+R_3)\,e^{(y-x)}\right] = [A3] + \frac{R_2}{R_1}\,e^{(y-x)}$$

$$C_2\left[(1-s_2) - (1-s_2-R_3)\,e^{-(y-x)}\,e^{-s_2(y-x)}\right]$$
$$+ D_2\left[(1+s_2)\,e^{-s_2(y-x)} - (1+s_2+R_3)\,e^{-(y-x)}\right] = [A3] + \frac{R_2}{R_1}\,e^{-(y-x)}$$

We solve the system

$$C_2 = \Big\{ [A3]\big[2s_2 e^{-s_2(y-x)} e^{-(y-x)} - (1+s_2+R_3)e^{-2(y-x)} + (1-s_2+R_3)\big]$$
$$+ \frac{R_2}{R_1}\big[(1+s_2)e^{-s_2(y-x)} - 2s_2 e^{-(y-x)} - (1-s_2)e^{-2(y-x)} e^{-s_2(y-x)}\big]\Big\}\frac{1}{\nabla_2}$$

$$D_2 = \Big\{ [A3]\big[2s_2 e^{-(y-x)} - (1+s_2-R_3)e^{-s_2(y-x)} + (1-s_2-R_3)e^{-2(y-x)} e^{-s_2(y-x)}\big]$$
$$+ \frac{R_2}{R_1}\big[(1+s_2)e^{-2(y-x)} - 2s_2 e^{-(y-x)} e^{-s_2(y-x)} - (1-s_2)\big]\Big\} \frac{1}{\nabla_2}$$

$$\nabla_2 = 8s_2 e^{-(y-x)} e^{-s_2(y-x)} - (1+s_2)\big[(1+s_2+R_3)e^{-2(y-x)} + (1+s_2-R_3)e^{-2s_2(y-x)}\big]$$
$$+(1-s_2)\big[(1-s_2+R_3) + (1-s_2-R_3)e^{-2(y-x)} e^{-2s_2(y-x)}\big]$$

For $m=\infty$, $\nabla_2 = (1-s_2)(1-s_2+R_3)$

We transform the w-condition utilizing the τ_{rz}-conditions

$$s_2(n_2-1)C_2 - s_2(n_2-1)D_2 e^{-s_2(y-x)} = K_1\big[s_3(n_3-1)C_3 e^{-s_3(z-y)} - s_3(n_3-1)D_3\big]$$

and with

$$K_2 = K_1 \frac{s_3(n_3-1)}{s_2(n_2-1)}$$

$$C_2 - D_2 e^{-s_2(y-x)} = K_2\big[C_3 e^{-s_3(z-y)} - D_3\big]$$

We replace C_2 and D_2 by their values

$$C_2 - D_2 e^{-s_2(y-x)} =$$
$$\Big\{[A3]\big[(1+s_2-R_3)e^{-2s_2(y-x)} + (1-s_2+R_3)$$
$$-(1+s_2+R_3)e^{-2(y-x)} - (1-s_2-R_3)e^{-2(y-x)} e^{-2s_2(y-x)}\big]$$
$$+\frac{R_2}{R_1}\big[2e^{-s_2(y-x)} - 2s_2 e^{-(y-x)} - 2e^{-2(y-x)} e^{-s_2(y-x)} + 2s_2 e^{-(y-x)} e^{-2s_2(y-x)}\big]\Big\}\frac{1}{\nabla_2}$$

$$= \Big\{[A3]\,\mathbb{O}_1 + \mathbb{O}_2\Big\}\frac{1}{\nabla_2} = K_2\big[C_3 e^{-s_3(z-y)} - D_3\big]$$

with

$$Q_3 = \frac{K_2 \nabla_2}{Q_1}$$

$$[A3] - Q_3\left[C_3 e^{-s_3(z-y)} - D_3\right] = -\frac{Q_2}{Q_1}$$

This relation becomes

If $s_3 > 1$

$$\left[1 + e^{-2(z-y)}\right] B_3 + \left[1 + \frac{s_3(n_3+\mu_3)}{(1+\mu_3)} e^{-(z-y)} e^{-s_3(z-y)} + Q_3\right] D_3 = -\frac{Q_2}{Q_1}$$

If $s_3 < 1$

$$\left[1 + (1-Q_3)\frac{1+\mu_3}{s_3(n_3+\mu_3)} e^{-(z-y)} e^{-s_3(z-y)}\right] B_3$$

$$+ \left[1 + (1-Q_3)\, e^{-2s_3(z-y)} + Q_3\right] D_3 = -\frac{Q_2}{Q_1}$$

Together with the τ_{rz}-condition

If $s_3 > 1$

$$\left[e^{-2(z-y)} - 1\right] B_3 + \left[\frac{s_3(n_3+\mu_3)}{(1+\mu_3)} e^{-(z-y)} e^{-s_3(z-y)} - s_3\right] D_3 = 0$$

If $s_3 < 1$

$$\left[\frac{1+\mu_3}{n_3+\mu_3} e^{-(z-y)} e^{-s_3(z-y)} - 1\right] B_3 + \left[s_3 e^{-2s_3(z-y)} - s_3\right] D_3 = 0$$

we solve the system

If $s_3 > 1$

$$B_3 = -\frac{Q_2}{Q_1}\,\frac{\frac{s_3(n_3+\mu_3)}{(1+\mu_3)} e^{-(z-y)} e^{-s_3(z-y)} - s_3}{\nabla_3}$$

$$D_3 = \frac{Q_2}{Q_1}\,\frac{e^{-2(z-y)} - 1}{\nabla_3}$$

$$\nabla_3 = (1 - s_3 + Q_3) - (1 + s_3 + Q_3)e^{-2(z-y)} + \frac{2s_3(n_3+\mu_3)}{(1+\mu_3)} e^{-(z-y)} e^{-s_3(z-y)}$$

if $s_3 < 1$

$$B_3 = -\frac{Q_2}{Q_1}\;\frac{s_3 e^{-2s_3(z-y)} - s_3}{\nabla_3}$$

$$D_3 = \frac{Q_2}{Q_1}\;\frac{\frac{1+\mu_3}{n_3+\mu_3}\, e^{-(z-y)}\, e^{-s_3(z-y)} - 1}{\nabla_3}$$

$$\nabla_3 = (1-s_3+Q_3) + (1+s_3-Q_3)e^{-2s_3(z-y)} - \frac{2(1+\mu_3)}{(n_3+\mu_3)}\, e^{-(z-y)}\, e^{-s_3(z-y)}$$

3. Relations for the parameters.

$$C_2 = \Big\{ [A3]\big[2s_2 e^{-s_2(y-x)} e^{-(y-x)} - (1+s_2+R_3)e^{-2(y-x)} + (1-s_2+R_3)\big] + \frac{R_2}{R_1}\big[(1+s_2)e^{-s_2(y-x)} - 2s_2 e^{-(y-x)} - (1-s_2)e^{-2(y-x)}\, e^{-s_2(y-x)}\big]\Big\}\frac{1}{\nabla_2}$$

$$D_2 = \Big\{ [A3]\big[2s_2 e^{-(y-x)} - (1+s_2-R_3)e^{-s_2(y-x)} + (1-s_2-R_3)e^{-2(y-x)}\, e^{-s_2(y-x)}\big] + \frac{R_2}{R_1}\big[(1+s_2)e^{-2(y-x)} - 2s_2 e^{-(y-x)} e^{-s_2(y-x)} - (1-s_2)\big]\Big\}\frac{1}{\nabla_2}$$

$$A_2 = \frac{1}{2}\Big\{ [A3] - (1+s_2)\,C_2 - (1-s_2)\,D_2 e^{-s_2(y-x)}\Big\}$$

$$B_2 = \frac{1}{2}\Big[-(1-s_2-R_3)\,C_2 e^{-s_2(y-x)} - (1+s_2+R_3)\,D_2 - \frac{R_2}{R_1}\Big]$$

The relations for the parameters A_1, B_1, C_1 and D_1 are given in § 1.3

Chapter 3. The four layered structure.

1. Supplementary boundary conditions.

Boundary conditions at the third interface ($z = H_1 + H_2 + H_3$):

$$\sigma_z: \quad A_3 + B_3 e^{-(z-y)} + C_3 + D_3 e^{-s_3(z-y)} =$$
$$A_4 e^{-(t-z)} + B_4 + C_4 e^{-s_4(t-z)} + D_4$$

$$\tau_{rz}: \quad A_3 - B_3 e^{-(z-y)} + s_3 C_3 - s_3 D_3 e^{-s_3(z-y)} =$$
$$A_4 e^{-(t-z)} - B_4 + s_4 C_4 e^{-s_4(t-z)} - s_4 D_4$$

$$w: \quad (1+\mu_3) A_3 - (1+\mu_3) B_3 e^{-(z-y)} + s_3(n_3+\mu_3) C_3 - s_3(n_3+\mu_3) D_3 e^{-s_3(z-y)} =$$
$$L_w \left[(1+\mu_4) A_4 e^{-(t-z)} - (1+\mu_4) B_4 + s_4(n_4+\mu_4) C_4 e^{-s_4(t-z)} - s_4(n_4+\mu_4) D_4 \right]$$

$$u: \quad (n_3+\mu_3) A_3 + (n_3+\mu_3) B_3 e^{-(z-y)} + (1+\mu_3) C_3 + (1+\mu_3) D_3 e^{-s_3(z-y)} =$$
$$L_u \left[(n_4+\mu_4) A_4 e^{-(t-z)} + (n_4+\mu_4) B_4 + (1+\mu_4) C_4 e^{-s_4(t-z)} + (1+\mu_4) D_4 \right]$$

Boundary conditions at the bottom ($z = H_1 + H_2 + H_3 + H_4$):

$w = 0$

$$(1+\mu_4) A_4 - (1+\mu_4) B_4 e^{-(t-z)} + s_4(n_4+\mu_4) C_4 - s_4(n_4+\mu_4) D_4 e^{-s_4(t-z)} = 0$$

if $s_4 > 1 \qquad C_4 = 0$

$$A_4 e^{-(t-z)} = B_4 e^{-2(t-z)} + \frac{s_4(n_4+\mu_4)}{(1+\mu_4)} D_4 e^{-(t-z)} e^{-s_4(t-z)}$$

if $s_4 < 1 \qquad A_4 = 0$

$$C_4 e^{-s_4(t-z)} = \frac{(1+\mu_4)}{s_4(n_4+\mu_4)} B_4 e^{-(t-z)} e^{-s_4(t-z)} + D_4 e^{-2s_4(t-z)}$$

2. Expression of the boundary conditions in matrixform.

2.1. At the third interface.

In the equations at the third interface, A_4, or C_4, is replaced by its value obtained from the fixed bottom condition.
We write the conditions at the third interface in matrixform

$$M_5\,(A_3\,B_3\,C_3\,D_3)^T = M_6\,(B_4\,D_4)^T$$

We invert M_5

$$(A_3\,B_3\,C_3\,D_3)^T = M_5^{-1}.\,M_6\,(B_4\,D_4)^T$$

$$M_5^{-1} = \frac{1}{2s_3(1-n_3)}\begin{pmatrix} s_3(1+\mu_3) & -s_3(n_3+\mu_3) & s_3 & -s_3 \\ s_3(1+\mu_3)e^{(z-y)} & s_3(n_3+\mu_3)e^{(z-y)} & -s_3e^{(z-y)} & -s_3e^{(z-y)} \\ -s_3(n_3+\mu_3) & (1+\mu_3) & -1 & s_3 \\ -s_3(n_3+\mu_3)e^{s_3(z-y)} & -(1+\mu_3)e^{s_3(z-y)} & e^{s_3(z-y)} & s_3e^{s_3(z-y)} \end{pmatrix}$$

If $s_4 > 1$

$$M_6 = \begin{pmatrix} 1+e^{-2(h-z)} & 1+\dfrac{s_4(n_4+\mu_4)}{(1+\mu_4)}e^{-(1+s_4)(h-z)} \\ -1+e^{-2(h-z)} & -s_4+\dfrac{s_4(n_4+\mu_4)}{(1+\mu_4)}e^{-(1+s_4)(h-z)} \\ L_1(1+\mu_4)\left[-1+e^{-2(h-z)}\right] & L_1s_4(n_4+\mu_4)\left[-1+e^{-(1+s_4)(h-z)}\right] \\ L_2(n_4+\mu_4)\left[1+e^{-2(h-z)}\right] & L_2(1+\mu_4)\left[1+\dfrac{s_4(n_4+\mu_4)^2}{(1+\mu_4)^2}e^{-(1+s_4)(h-z)}\right] \end{pmatrix}$$

If $s_4 < 1$

$$M_6 \begin{pmatrix} 1+\dfrac{1+\mu_4}{s_4(n_4+\mu_4)}e^{-(1+s_4)(h-z)} & 1+e^{-2s_4(h-z)} \\ -1+\dfrac{1+\mu_4}{(n_4+\mu_4)}e^{-(1+s_4)(h-z)} & -s_4+e^{-2s_4(h-z)}.s_4 \\ L_1(1+\mu_4)\left[-1+e^{-(1+s_4)(h-z)}\right] & L_1s_4'(n_4+\mu_4)\left[-1+e^{-2s_4(h-z)}\right] \\ L_2(n_4+\mu_4)\left[1+\dfrac{(1+\mu_4)^2}{s_4(n_4+\mu_4)^2}.e^{-(1+s_4)(h-z)}\right] & L_2(1+\mu_4)\left[1+e^{-2s_4(h-z)}\right] \end{pmatrix}$$

We write A_3, B_3, C_3 and D_3 in function of B_4 and D_4.

$$A_3 = \frac{\sum M_5(1i) \cdot M_6(i1) \cdot B_4 + \sum M_5(1i) \cdot M_6(i2) D_4}{2 s_3 (1-n_3)}$$

$$B_3 = \frac{\sum M_5(2i) \cdot M_6(i1) \cdot B_4 + \sum M_5(2i) \cdot M_6(i2) D_4}{2 s_3 (1-n_3)} \cdot e^{(z-y)}$$

$$C_3 = \frac{\sum M_5(3i) \cdot M_6(i1) \cdot B_4 + \sum M_5(3i) \cdot M_6(i2) D_4}{2 s_3 (1-n_3)}$$

$$D_3 = \frac{\sum M_5(4i) \cdot M_6(i1) \cdot B_4 + \sum M_5(4i) \cdot M_6(i2) D_4}{2 s_3 (1-n_3)} e^{s_3(z-y)}$$

where $M_5(i,j)$ are the constants in M_5^{-1}.

We write

$$P_{j1} = \sum M_5(ji) M_6(i1)$$

$$P_{j2} = \sum M_5(ji) M_6(i2)$$

so that

$$\begin{pmatrix} A_3 \\ B_3 \\ C_3 \\ D_3 \end{pmatrix} = \frac{1}{2 s_3 (1-n_3)} \begin{pmatrix} P_{11} & P_{12} \\ P_{21} e^{(z-y)} & P_{22} e^{(z-y)} \\ P_{31} & P_{32} \\ P_{41} e^{s_3(z-y)} & P_{42} e^{s_3(z-y)} \end{pmatrix} \cdot \begin{vmatrix} B_4 \\ D_4 \end{vmatrix}$$

2.2. At the surface

Adding and substracting the surface conditions we obtain

$$2A_1 e^{-x} = 1-(1+s_1)C_1 e^{-s_1 x} - (1-s_1)D_1$$

$$2B_1 = 1-(1-s_1)C_1 e^{-s_1 x} - (1+s_1)D_1$$

2.3. At the first interface.

We add and substract the first two conditions

$$2A_1 + (1+s_1)C_1 + (1-s_1)D_1 e^{-s_1 x} = A_2 e^{-(y-x)} + B_2 + C_2 e^{-s_2(y-x)} + D_2$$

$$= [A2]$$

$$2B_1 e^{-x} + (1-s_1)C_1 + (1+s_1)D_1 e^{-s_1 x} = [A2]$$

We replace A_1 and B_1 by their values obtained from the surface conditions

$$(1+s_1)C_1\left[1-e^{x}e^{-s_1 x}\right] + (1-s_1)D_1\left[e^{-s_1 x} - e^{x}\right] = [A2]-e^{x}$$

$$(1-s_1)C_1\left[1-e^{-x}e^{-s_1 x}\right] + (1+s_1)D_1\left[e^{-s_1 x} - e^{-x}\right] = [A2]-e^{-x}$$

We solve the system

$$C_1 = \Big\{[A2]\cdot\left[2s_1 e^{-s_1 x} - (1+s_1)e^{-x} + (1-s_1)e^{x}\right]$$

$$+\left[2s_1 - (1+s_1)e^{x}e^{-s_1 x} + (1-s_1)e^{-x}e^{-s_1 x}\right]\Big\}\cdot\frac{1}{e^{x}\cdot\nabla_1}$$

$$= \Big\{[A2]\cdot\left[2s_1 e^{-x}\cdot e^{-s_1 x} - (1+s_1)e^{-2x} + (1-s_1)\right]$$

$$+\left[2s_1 e^{-x} - (1+s_1)e^{-s_1 x} + (1-s_1)e^{-2x}\cdot e^{-s_1 x}\right]\Big\}\cdot\frac{1}{\nabla_1}$$

The positive exponent e^{x} has disappeared.

$$D_1 = \Big\{[A2]\cdot\left[2s_1 e^{-x} - (1+s_1)e^{-s_1 x} + (1-s_1)e^{-2x}e^{-s_1 x}\right]$$

$$+\left[2s_1 e^{-x}e^{-s_1 x} - (1+s_1)e^{-2x} + (1-s_1)\right]\Big\}\cdot\frac{1}{\nabla_1}$$

$$\nabla_1 = 8s_1 e^{-x}e^{-s_1 x} - (1+s_1)^2\left[e^{-2x} + e^{-2s_1 x}\right] + (1-s_1)^2\left[1 + e^{-2x}\cdot e^{-2s_1 x}\right]$$

For $m = \infty$ $\qquad \nabla_1 = (1-s_1)^2$

We transform the w-condition utilizing the τ_{rz}-conditions

$$s_1(n_1-1)C_1 - s_1(n_1-1)D_1 e^{-s_1x} = F_1\left[s_2(n_2-1)C_2 e^{-s_2(y-x)} - s_2(n_2-1)D_2\right]$$

$$F = F_1 \frac{s_2(n_2-1)}{s_1(n_1-1)}$$

$$C_1 - D_1 e^{-s_1x} = FC_2 e^{-s_2(y-x)} - FD_2$$

We replace C_1 and D_1 by their values

$$C_1 - D_1 e^{-s_1x} =$$

$$\left\{[A2]\left[(1-s_1) - (1+s_1)e^{-2x} + (1+s_1)e^{-2s_1x} - (1-s_1)e^{-2x}e^{-2s_1x}\right] + \left[2s_1e^{-x}(1-e^{-2s_1x}) + 2e^{-2x}e^{-s_1x} - 2e^{-s_1x}\right]\right\} \cdot \frac{1}{\nabla_1}$$

$$= \left\{[A2] \cdot R_1 + R_2\right\} \cdot \frac{1}{\nabla_1}$$

For $m = \infty \qquad R_1 = (1-s_1) \quad , \quad R_2 = 0$

$$\left\{[A2] \cdot R_1 + R_2\right\} \cdot \frac{1}{\nabla_1} = FC_2 e^{-s_2(y-x)} - FD_2$$

$$[A2] - \frac{F\nabla_1}{R_1}\left[C_2 e^{-s_2(y-x)} - D_2\right] = -\frac{R_2}{R_1}$$

Writing $R_3 = \dfrac{F\nabla_1}{R_1}$, we obtain the system

$$A_2 e^{-(y-x)} + B_2 + C_2 e^{-s_2(y-x)} + D_2 - R_3 C_2 e^{-s_2(y-x)} + R_3 D_2 = -\frac{R_2}{R_1}$$

$$A_2 e^{-(y-x)} - B_2 + s_2 C_2 e^{-s_2(y-x)} - s_2 D_2 = 0$$

and by adding and substracting

$$2A_2 e^{-(y-x)} = -(1+s_2-R_3)C_2 e^{-s_2(y-x)} - (1-s_2+R_3)D_2 - \frac{R_2}{R_1}$$

$$2B_2 = -(1-s_2-R_3)C_2 e^{-s_2(y-x)} - (1+s_2+R_3)D_2 - \frac{R_2}{R_1}$$

2.4. At the second interface.

We add and substract the first two conditions

$$2A_2 + (1+s_2)C_2 + (1-s_2)D_2 e^{-s_2(y-x)} = A_3 e^{-(z-y)} + B_3 + C_3 e^{-s_3(z-y)} + D_3$$
$$= [A3]$$

$$2B_2 e^{-(y-x)} + (1-s_2)C_2 + (1+s_2)D_2 e^{-s_2(y-x)} = [A3]$$

We replace A_2 and B_2 by their values obtained from the first interface conditions

$$C_2\left[(1+s_2) - (1+s_2-R_3)e^{(y-x)}e^{-s_2(y-x)}\right]$$
$$+ D_2\left[(1-s_2)e^{-s_2(y-x)} - (1-s_2+R_3)e^{(y-x)}\right] = [A3] + \frac{R_2}{R_1} e^{(y-x)}$$

$$C_2\left[(1-s_2) - (1-s_2-R_3)e^{-(y-x)}e^{-s_2(y-x)}\right]$$
$$+ D_2\left[(1+s_2)e^{-s_2(y-x)} - (1+s_2+R_3)e^{-(y-x)}\right] = [A3] + \frac{R_2}{R_1} e^{-(y-x)}$$

We solve the system

$$C_2 = \Big\{[A3]\left[2s_2 e^{-s_2(y-x)} - (1+s_2+R_3)e^{-(y-x)} + (1-s_2+R_3)e^{(y-x)}\right]$$
$$+ \frac{R_2}{R_1}\left[(1+s_2)e^{(y-x)}e^{-s_2(y-x)} - 2s_2 - (1-s_2)e^{-(y-x)}e^{-s_2(y-x)}\right]\Big\}\frac{1}{e^{(y-x)}\cdot\nabla_2}$$

$$= \Big\{[A3]\left[2s_2 e^{-(y-x)}e^{-s_2(y-x)} - (1+s_2+R_3)e^{-2(y-x)} + (1-s_2+R_3)\right]$$
$$+ \frac{R_2}{R_1}\left[(1+s_2)e^{-s_2(y-x)} - 2s_2 e^{-(y-x)} - (1-s_2)e^{-2(y-x)}e^{-s_2(y-x)}\right]\Big\}\frac{1}{\nabla_2}$$

The positive exponent $e^{(y-x)}$ has disappeared.

$$D_2 = \Big\{[A3]\left[2s_2 e^{-(y-x)} - (1+s_2-R_3)e^{-s_2(y-x)} + (1-s_2-R_3)e^{-2(y-x)}e^{-s_2(y-x)}\right]$$
$$+ \frac{R_2}{R_1}\left[(1+s_2)e^{-2(y-x)} - 2s_2 e^{-(y-x)}e^{-s_2(y-x)} - (1-s_2)\right]\Big\}\cdot\frac{1}{\nabla_2}$$

$$\nabla_2 = 8s_2 e^{-(y-x)}e^{-s_2(y-x)} - (1+s_2)\left[(1+s_2+R_3)e^{-2(y-x)} + (1+s_2-R_3)e^{-2s_2(y-x)}\right]$$
$$+ (1-s_2)\left[(1-s_2+R_3) + (1-s_2-R_3)e^{-2(y-x)}\cdot e^{-2s_2(y-x)}\right]$$

For $m=\infty$, $\nabla_2 = (1-s_2)(1-s_2+R_3)$

We transform the w-condition utilizing the τ_{rz}-conditions

$$s_2(n_2-1)C_2 - s_2(n_2-1)D_2 e^{-s_2(y-x)} = k_1\left[s_3(n_3-1)C_3 e^{-s_3(z-y)} - s_3(n_3-1)D_3\right]$$

$$k = k_1 \frac{s_3(n_3-1)}{s_2(n_2-1)}$$

$$C_2 - D_2 e^{-s_2(y-x)} = k C_3 e^{-s_3(z-y)} - k D_3$$

We replace C_2 and D_2 by their values

$$C_2 - D_2 e^{-s_2(y-x)} =$$

$$\Big\{[A3]\Big[(1+s_2-R_3)e^{-2s_2(y-x)} + (1-s_2+R_3) - (1+s_2+R_3)e^{-2(y-x)} - (1-s_2-R_3)e^{-2(y-x)}e^{-2s_2(y-x)}\Big] + \frac{R_2}{R_1}\Big[2e^{-s_2(y-x)} - 2s_2 e^{-(y-x)} - 2e^{-2(y-x)}e^{-s_2(y-x)} + 2s_2 e^{-(y-x)}e^{-2s_2(y-x)}\Big]\Big\}\cdot\frac{1}{\nabla_2}$$

$$= \Big\{[A3]\,Q_1 + Q_2\Big\}\frac{1}{\nabla_2}$$

For $m = \infty$, $\quad Q_1 = (1-s_2+R_3)$, $\quad Q_2 = 0$

Writing $Q_3 = \dfrac{k\nabla_2}{Q_1}$, we obtain the system

$$A_3 e^{-(z-y)} + B_3 + C_3 e^{-s_3(z-y)} + D_3 - Q_3 C_3 e^{-s_3(z-y)} + Q_3 D_3 = -\frac{Q_2}{Q_1}$$

$$A_3 e^{-(z-y)} - B_3 + s_3 C_3 e^{-s_3(z-y)} - s_3 D_3 = 0$$

and by adding and substracting

$$2A_3 e^{-(z-y)} + (1+s_3-Q_3)C_3 e^{-s_3(z-y)} + (1-s_3+Q_3)D_3 = -\frac{Q_2}{Q_1}$$

$$2B_3 + (1-s_3-Q_3)C_3 e^{-s_3(z-y)} + (1+s_3+Q_3)D_3 = -\frac{Q_2}{Q_1}$$

3. Resolution of the system of boundary conditions.

We have from the boundary conditions at the third interface (§ 2.1)

$$A_3 = \frac{1}{2s_3(1-n_3)}\left[P_{11}.B_4 + P_{12}.D_4\right]$$

$$B_3 = \frac{1}{2s_3(1-n_3)}\left[P_{21}.B_4 + P_{22}D_4\right]e^{(z-y)}$$

$$C_3 = \frac{1}{2s_3(1-n_3)}\left[P_{31}.B_4 + P_{32}D_4\right]$$

$$D_3 = \frac{1}{2s_3(1-n_3)}\left[P_{41}.B_4 + P_{42}D_4\right]e^{s_3(z-y)}$$

We replace A_3, B_3, C_3 and D_3 in the last equations of § 2.4

$$\left[2P_{11}e^{-(z-y)} + (1+s_3-\Theta_3)P_{31}e^{-s_3(z-y)} + (1-s_3+\Theta_3)P_{41}e^{s_3(z-y)}\right]B_4$$
$$+\left[2P_{12}e^{-(z-y)} + (1+s_3-\Theta_3)P_{32}e^{-s_3(z-y)} + (1-s_3+\Theta_3)P_{42}e^{s_3(z-y)}\right]D_4$$
$$= -2s_3(1-n_3)\frac{\Theta_2}{\Theta_1}$$

$$\left[2P_{21}e^{(z-y)} + (1-s_3-\Theta_3)P_{31}e^{-s_3(z-y)} + (1+s_3+\Theta_3)P_{41}e^{s_3(z-y)}\right]B_4$$
$$+\left[2P_{22}e^{(z-y)} + (1-s_3-\Theta_3)P_{32}e^{-s_3(z-y)} + (1+s_3+\Theta_3)P_{42}e^{s_3(z-y)}\right]D_4$$
$$= -2s_3(1-n_3)\frac{\Theta_2}{\Theta_1}$$

We solve the system in B_4 and D_4

$$B_4 = -2s_3(1-n_3)\frac{\Theta_2}{\Theta_1}\Big[-2P_{12}e^{-(z-y)} + 2P_{22}e^{(z-y)}$$
$$-2s_3P_{32}e^{-s_3(z-y)} + 2s_3P_{42}e^{s_3(z-y)}\Big]\frac{1}{2e^{(z-y)}e^{s_3(z-y)}\nabla_3}$$

$$B_4 = 2s_3(1-n_3)\frac{\Phi_2}{\Phi_1}\Big[P_{12}e^{-2(z-y)}e^{-s_3(z-y)} - P_{22}e^{-s_3(z-y)}$$
$$+ s_3 \cdot P_{32}e^{-(z-y)}e^{-2s_3(z-y)} - s_3 P_{42}e^{-(z-y)}\Big]\cdot\frac{1}{\nabla_3}$$

The positive exponents $e^{(z-y)}$, $e^{s_3(z-y)}$ and $e^{(z-y)}\cdot e^{s_3(z-y)}$ have disappeared.

$$D_4 = 2s_3(1-n_3)\frac{\Phi_2}{\Phi_1}\Big[P_{21}e^{-s_3(z-y)} - P_{11}e^{-2(z-y)}\cdot e^{-s_3(z-y)}$$
$$+ s_3 P_{41}e^{-(z-y)} - s_3 P_{31}e^{-(z-y)}e^{-2s_3(z-y)}\Big]\cdot\frac{1}{\nabla_3}$$

$$\nabla_3 = 2\Big[(P_{11}\cdot P_{22} - P_{12}\cdot P_{21}) + s_3(P_{31}\cdot P_{42} - P_{32}\cdot P_{41})\Big]e^{-(z-y)}e^{-s_3(z-y)}$$
$$+ (1+s_3+\Phi_3)(P_{11}P_{42} - P_{12}P_{41})\,e^{-2(z-y)}$$
$$+ (1+s_3-\Phi_3)(P_{31}P_{22} - P_{32}P_{21})\,e^{-2s_3(z-y)}$$
$$+ (1-s_3-\Phi_3)(P_{11}P_{32} - P_{12}P_{31})\,e^{-2(z-y)}\cdot e^{-2s_3(z-y)}$$
$$+ (1-s_3+\Phi_3)(P_{22}P_{41} - P_{21}P_{42})$$

For $m=\infty$ $$\nabla_3 = \big[(1-s_3) + k(1-s_2)\big](P_{22}P_{41} - P_{21}P_{42})$$

The numerators of B_4 and D_4 tend both to zero and the denominator tends to a constant value.

4. Values of the parameters A_i, D_i.

4.1. Values of the parameters A_3, B_3, C_3, D_3.

The values of the parameters A_3, B_3, C_3 and D_3 are obtained from the boundary conditions at the third interface in which B_4 and D_4 are replaced by their values from § 3.

$$A_3 = \left[P_{11} B_4 + P_{12} D_4 \right] \cdot \frac{1}{2s_3(1-n_3)}$$

$$= \frac{Q_2}{Q_1} \Big[\left(P_{12} \cdot P_{21} - P_{11} \cdot P_{22} \right) \cdot e^{-s_3(z-y)} + s_3 \left(P_{11} P_{32} - P_{12} P_{31} \right) e^{-(z-y)} e^{-2s_3(z-y)}$$
$$+ s_3 \left(P_{12} \cdot P_{41} - P_{11} \cdot P_{42} \right) e^{-(z-y)} \Big] \frac{1}{\nabla_3}$$

$$B_3 = \left[P_{21} B_4 + P_{22} D_4 \right] \cdot \frac{e^{(z-y)}}{2s_3(1-n_3)}$$

$$= \frac{Q_2}{Q_1} \Big[\left(P_{12} \cdot P_{21} - P_{11} \cdot P_{22} \right) e^{-2(z-y)} \cdot e^{-s_3(z-y)}$$
$$+ s_3 \left(P_{21} \cdot P_{32} - P_{22} \cdot P_{31} \right) e^{-(z-y)} e^{-2s_3(z-y)} + s_3 \left(P_{22} P_{41} - P_{21} P_{42} \right) e^{-(z-y)} \Big] \frac{e^{(z-y)}}{\nabla_3}$$

$$= \frac{Q_2}{Q_1} \Big[\left(P_{12} \cdot P_{21} - P_{11} \cdot P_{22} \right) e^{-(z-y)} e^{-s_3(z-y)}$$
$$+ s_3 \left(P_{21} \cdot P_{32} - P_{22} \cdot P_{31} \right) e^{-2s_3(z-y)} + s_3 \left(P_{22} \cdot P_{41} - P_{21} \cdot P_{42} \right) \Big] \frac{1}{\nabla_3}$$

The positive exponent $e^{(z-y)}$ has disappeared. Although the presence of the constant $s_3(P_{22}P_{41} - P_{21}P_{42})$ the numerator converges to zero because of the factor Q_2.

$$C_3 = \left[P_{31} B_4 + P_{32} D_4 \right] \frac{1}{2s_3(1-n_3)}$$

$$= \frac{Q_2}{Q_1} \Big[\left(P_{31} \cdot P_{12} - P_{32} \cdot P_{11} \right) e^{-2(z-y)} \cdot e^{-s_3(z-y)} + \left(P_{32} \cdot P_{21} - P_{31} P_{22} \right) e^{-s_3(z-y)}$$
$$+ s_3 \left(P_{32} P_{41} - P_{31 42} \right) e^{-(z-y)} \Big] \frac{1}{\nabla_3}$$

$$D_3 = \left[P_{41}.B_4 + P_{42}.D_4 \right] \frac{e^{s_3(z-y)}}{2s_3(1-h_3)}$$

$$= \frac{\mathbb{O}_2}{\mathbb{O}_1} \left[(P_{41}.P_{12} - P_{11}.P_{42})\, e^{-2(z-y)} + (P_{42}.P_{21} - P_{41}.P_{22}) + s_3 (P_{41}.P_{32} - P_{31}.P_{42})\, e^{-(z-y)} e^{-s_3(z-y)} \right] \frac{1}{\nabla_3}$$

4.2. Values of the parameters A_2, B_2, C_2, D_2.

The values of C_2 and D_2 are obtained from the relations established in § 2.4.

$$C_2 = \frac{1}{\nabla_2} \Big\{ \left[2s_2 e^{-(y-x)} . e^{-s_2(y-x)} - (1+s_2+R_3)\, e^{-2(y-x)} + (1-s_2+R_3) \right] . \left[A_3 e^{-(z-y)} + B_3 + C_3 e^{-s_3(z-y)} + D_3 \right] + \frac{R_2}{R_1} \left[(1+s_2)\, e^{-s_2(y-x)} - 2s_2 e^{-(y-x)} - (1-s_2)\, e^{-2(y-x)} e^{-s_2(y-x)} \right] \Big\}$$

$$D_2 = \frac{1}{\nabla_2} \Big\{ \left[2s_2 e^{-(y-x)} - (1+s_2-R_3)\, e^{-s_2(y-x)} + (1-s_2-R_3)\, e^{-2(y-x)} . e^{-s_2(y-x)} \right] \left[A_3 e^{-(z-y)} + B_3 + C_3\, e^{-s_3(z-y)} + D_3 \right] + \frac{R_2}{R_1} \left[(1+s_1) e^{-2(y-x)} - 2s_2 e^{-(y-x)} e^{-s_2(y-x)} - (1-s_2) \right] \Big\}$$

The numerator in D_2 converges because of the presence of the factor R_2.

The value of A_2 is obtained from the relation established in § 2.4.

$$A_2 = \frac{1}{2} \left[A_3\, e^{-(z-y)} + B_3 + C_3 e^{-s_3(z-y)} + D_3 - (1+s_2)\, C_2 - (1-s_2) D_2 e^{s_2(y-x)} \right]$$

The value of B_2 is obtained from the last relation of § 2.3.

$$B_2 = -\frac{1}{2} \left[(1-s_2-R_3)\, C_2\, e^{-s_2(y-x)} + (1+s_2+R_3)\, D_2 + \frac{R_2}{R_1} \right]$$

4.3. Values of the parameters A_1, B_1, C_1, D_1.

The values of the parameters A_1 and C_1 are obtained from the relations established in § 2.3.

$$C_1 = \frac{1}{\nabla_1} \Big\{ \left[2s_1 e^{-x} e^{-s_1 x} - (1+s_1) e^{-2x} + (1-s_1) \right] \cdot \left[A_2 e^{-(y-x)} + B_2 + C_2 e^{-s_2(y-x)} + D_2 \right] + \left[2s_1 e^{-x} - (1+s_1) e^{-s_1 x} + (1-s_1) e^{-2x} e^{-s_1 x} \right] \Big\}$$

For the determination of A_1, we need the value of $D_1 e^{-s_1 x}$.

$$D_1 e^{-s_1 x} = \frac{1}{\nabla_1} \Big\{ \left[2s_1 e^{-x} e^{-s_1 x} - (1+s_1) e^{-2s_1 x} + (1-s_1) e^{-2x} e^{-2s_1 x} \right] \left[A_2 e^{-(y-x)} + B_2 + C_2 e^{-s_2(y-x)} + D_2 \right] + \left[2s_1 e^{-x} \cdot e^{-2s_1 x} - (1+s_1) e^{-2x} e^{-s_1 x} + (1-s_1) e^{-s_1 x} \right]$$

$$A_1 = \frac{1}{2} \left[A_2 e^{-(y-x)} + B_2 + C_2 e^{-s_2(y-x)} + D_2 - (1+s_1) C_1 - (1-s_1) D_1 e^{-s_1 x} \right]$$

The values of B_1 and D_1 are obtained from the surface conditions.

$$B_1 = \frac{1}{s_1 - 1} \left[s_1 - (1+s_1) A_1 e^{-x} - 2s_1 C_1 e^{-s_1 x} \right]$$

$$D_1 = -\frac{1}{s_1 - 1} \left[1 - 2A_1 e^{-x} - (1+s_1) C_1 e^{-s_1 x} \right]$$

5. Relations for the stresses and the displacements.

The relations for the stresses and the displacements are completely the same as those developped in appendix 3, by replacing the parameters A_i, D_i by their adequate values.

The relation for the verticale displacement is again undeterminated at the origin (m = 0). The problem is solved in exactly the same way as developped in appendix 2 (§ 4.).

APPENDIX 5

EXPLANATORY NOTICE

****

Four-layered System Program :

This program is available in two versions:

- EXECUTABLE version, in which the executable program is made up of only one block with automatic loading.
- SOURCE version, in which all controls, data, and all modules are in separated files.

Floppy disk contents :

```
In following text, x means  A = ANISOTROPIC
                            I = ISOTROPIC
                   y means  P = PARTIAL
                            S = SLIP
```

Each floppy disk contains following main files:

```
AUTOEXEC.BAT    ----> automatic program loading
LOGO.BAT        ----> display of introduction logo
FLxyLO.TXT      ----> introduction text on screen
FLxyNO.LOG      ----> introduction text for printer
FLxyNO.TXT      ----> this notice
```

*EXECUTABLE floppy disk

This disk contains the following files in addition to the main files:

```
FLxy.EXE      ---> executable program
FLxy.DAT      ---> data file for demonstration
FLxy.LST      ---> result file for demonstration
```

*SOURCE floppy disk

This floppy disk contains the following files in addition to the main files:

```
FLxy.VER      ---> revision
FLxy1.FOR     ---> main module
FLxy2.FOR     ---> subroutines     DOMEC
                                   DOTRA
                                   POCAL
                                   POINT
                                   PAS
                                   CHECH
                                   ERROR
FLxy3.FOR     ---> subroutines     ECHDE
                                   VINIT
                                   ZERO
                                   FINIT
FLxy4.FOR     ---> subroutines     P4442
                                   PCT22
                                   PCT42
                                   SOM42
                                   SOM22
                                   P4444
                                   P2442
                                   CONST
```

FLxy5.FOR	---> subroutines	BESJ1 BJOJ2 SURFA COUC1 COUC2 COUC3 COUC4
FLxy6.FOR	---> subroutines	TITRE MODIF SIMPS
FLxy7.FOR	---> subroutines	COMSU FONC CCOU1 FONC1 FONC2 FONC3 FONC4 FONC5 FONC6 ECHEF
FLxy8.FOR	---> subroutines	IMDON IMRES
FLxy9.FOR	---> subroutines	MENU ECDON LEDON AFDMC AFDTR AFPOC

Configuration:

IBM-PC (G,XT,AT) with at least 256KB, 1 ou 2 diskette drives, 80 col. screen (monochrome or color), math. coprocessor, matrix printer.

Running of the program:

In the next presentation <ENTER> means action of key <--- .
Insert the EXECUTABLE disk into drive B.
Load DOS, if necessary, then type DIR B: and PATH A:\
FLB> FLxy <ENTER>

Printing of the results:

on screen   : FLB> type FLxy.LST <ENTER>
on printer : FLB> type FLxy.LST >prn <ENTER>

Consultation of this notice:

on screen   : FLB> type FLxyNO.TXT <ENTER>
on printer : FLB> type FLxyNO.TXT >prn <ENTER>

Disk preparation for normal running:

Format a system disk with COMMAND.COM, and files needed for running AUTOEXEC.BAT.
Preparation of CONFIG.SYS with files=10, device=ansi, buffers=10.

Alterations in source files:

By text editor EDLIN or any other text editor, program statements, may be altered.

Commands for EDLIN are:

```
     FLB> edlin xxxxn.for  <ENTER>
            where ---> file name to be modified  (xxxxn.for)

     nD      --> erase line number n
     n       --> displays line n for alteration, type the
                 correct statement.
     n1,n2L  --> displays lines between number n1 and number n2
     E       --> ends session and returns to DOS
```

Compilation:

After alteration, the new version of the module has to be compiled.

Insert Professional fortran compiler into drive A.

```
FLA> b:  <ENTER>
FLB> path a:\  <ENTER>
FLB> profort  xxxx/li  <ENTER>
```

Linkage:

After correct compilation has taken place, a new executable program has to be created.
Insert diskette with FORTRAN libraries into drive A.

```
FLB> link FLxy1+FLxy2+...,FLxy,CON:;   <ENTER>
```

Note :

To obtain introduction logo on printer

```
FLB> type FLxyNO.LOG >PRN   <ENTER>
```

Presentation of different possible screens:
===

FD-USAE-vers. 2.00 1986

MAIN MENU scren :
================

```
Strains and stresses
      in                                                              USAE-2
a four-layered system.
*******************************************************************************

 1. Data retrieval in a file
 2. Data saving in a file
 3. Screen displaying and/or alteration of system data
 4. Screen displaying and/or alteration of traffic data
 5. Screen displaying and/or alteration of computation coordinates
 6. Input of intermediate depths
 7. Program start




Your choice ----- 0 for stop -- :
---------------------------------------------------------------------
```

Screen of CHOICE 1:
===================

```
Strains and stresses
      in                                                              USAE-2
a four-layered system.
*******************************************************************************

Data retrieval in a file.
------------------------

A FLxy.DAT file contains base data.
The user can define another file whose name has to be written in
8 characters (format XXXX.DAT).

Name of chosen file or FLxy.DAT :
---------------------------------------------------------------------
```

****

Screen of CHOICE 2:
==================

Strains and stresses
in USAE-2
a four-layered system.
****

Data saving in a file.

A FLxy.DAT file contains base data.
The user can define another file whose name has to be written in
8 characters (format XXXX.DAT).

Name of chosen file or FLxy.DAT :

Screen of CHOICE 3:
==================

Strains and stresses
in USAE-2
a four-layered system.
****

System data.
============

Layer	Modulus	Poisson's r.	Thickness	Friction ra.
1.	400000.0	0.16	20.0	
				1.0000
2.	100000.0	0.25	20.0	
				1.0000
3.	10000.0	0.50	30.0	
				1.0000
4.	1000.0	0.50	9000.0	

For return =1 or alter. =2 ---- :

****

Screen of CHOICE 4:
===================

Strains and stresses
in USAE-2
a four-layered system.
****

Number of circular loads = 2

Radius	Pressure	Dist. x	Dist. y
11.450	7.900	0.000	0.000
11.450	7.900	34.350	0.000

For continuation = 1 or alter. = 2 ---- :

Screen of CHOICE 5:
===================

Strains and stresses
in USAE-2
a four-layered system.
****

Number of computation coordinates = 2

x=	y=
0.000	0.000
17.175	0.000

For continuation= 1 or alter. = 2 ---- :

Screen of CHOICE 6:
===================

Strains and stresses
in USAE-2
a four-layered system.
****

Positions of stress computations in depth out of interfaces.
(max 22)
number of positions :

****

Screen of CHOICE 7:
==================

Strains and stresses
in USAE-2
a four-layered system.
****

initial computation interval (0.1 is generaly small enough) :

Next screen:
============

Strains and stresses
in USAE-2
a four-layered system.
****

choice of scale :
****

allowed choices :
1 ---- thickness of 1rst layer
2 ---- thickness of 2 first layers
3 ---- thickness of 3 first layers
4 ---- thickness of 4 layers
5 ---- load radius

suggested solution : 5
your choice :

Screen for execution:
=====================

Strains and stresses
in USAE-2
a four-layered system.
****

computation start..... be patient!!

m = 0.10
be even more patient...!!

****

Screen for results:
===================

Strains and stresses
in USAE-2
a four-layered system.
****

A file FLxy.LST contains base results.
The user can define another file whose name has to be written in
8 characters (format XXXX.DAT).

Name of chosen file or FLxy.LST :

****

Sample of results (FLxy.LST)

Four-layered system program isotropic - partial friction

USAE-2

departement des constructions
isic
av.de l'hopital, 27 h
7000 mons belgique

##

Mechanical data
****

Young's modulus	P.'s ratio	thickness	friction rat
400000.0	0.16	20.000	
			1.00
100000.0	0.25	20.000	
			1.00
10000.0	0.50	30.000	
			1.00
1000.0	0.50	9000.000	

traffic data
****

load	radius	pressure	x	y
1	11.450	7.900	0.000	0.000
2	11.450	7.900	34.350	0.000

!!

position 1 x = 0.000 y = 0.000

depth	layer	sx	sy	sz	tyz	txz	txy
0.000	1 !!	10.1660	11.6793	7.9000	0.0000	0.0000	0.0000
20.000	1 !!	-3.4792	-4.5018	1.4988	0.0000	-0.4167	0.0000
20.000	2 !!	-0.5790	-0.8163	1.4989	0.0000	-0.4166	0.0000
40.000	2 !!	-2.2733	-2.5873	0.1873	0.0000	-0.0611	0.0000
40.000	3 !!	-0.1736	-0.1997	0.1873	0.0000	-0.0611	0.0000
70.000	3 !!	-0.3907	-0.4035	0.0540	0.0000	-0.0046	0.0000
70.000	4 !!	0.0095	0.0082	0.0540	0.0000	-0.0046	0.0000
9070.000	4 !!	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

depth	layer	s1	s2	s3	eps1	eps2	eps3
0.000	1 !!	11.6793	10.1660	7.9000	0.2197E-04	0.1758E-04	0.1101E-04
20.000	1 !!	1.5335	-3.5138	-4.5018	0.7040E-05	-.7597E-05	-.1046E-04
20.000	2 !!	1.5793	-0.6595	-0.8163	0.1948E-04	-.8502E-05	-.1046E-04
40.000	2 !!	0.1888	-2.2749	-2.5873	0.1404E-04	-.1675E-04	-.2066E-04
40.000	3 !!	0.1973	-0.1836	-0.1997	0.3890E-04	-.1824E-04	-.2066E-04
70.000	3 !!	0.0540	-0.3908	-0.4035	0.4512E-04	-.2160E-04	-.2352E-04
70.000	4 !!	0.0545	0.0091	0.0082	0.4582E-04	-.2230E-04	-.2352E-04
9070.000	4 !!	0.0000	0.0000	0.0000	0.0000E+00	0.0000E+00	0.0000E+00

depth	layer	u(x)	v(y)	w(z)	ex	ey	ez
0.000	1 !!	-.2184E-03	-.1909E-10	-.9046E-02	0.1758E-04	0.2197E-04	0.1101E-04
20.000	1 !!	0.9885E-04	0.8642E-11	-.8835E-02	-.7497E-05	-.1046E-04	0.6939E-05
20.000	2 !!	0.9885E-04	0.8642E-11	-.8835E-02	-.7497E-05	-.1046E-04	0.1848E-04
40.000	2 !!	0.3017E-03	0.2637E-10	-.8561E-02	-.1673E-04	-.2066E-04	0.1402E-04
40.000	3 !!	0.3017E-03	0.2637E-10	-.8561E-02	-.1673E-04	-.2066E-04	0.3739E-04
70.000	3 !!	0.3844E-03	0.3360E-10	-.7432E-02	-.2159E-04	-.2352E-04	0.4511E-04
70.000	4 !!	0.3844E-03	0.3360E-10	-.7432E-02	-.2159E-04	-.2352E-04	0.4511E-04
9070.000	4 !!	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

##

position 2 x = 17.175 y = 0.000

depth	layer	sx	sy	sz	tyz	txz	txy
0.000	1 ##	3.0499	8.8259	0.0000	0.0000	0.0000	0.0000
20.000	1 ##	-2.0893	-4.3575	1.1255	0.0000	0.0000	0.0000
20.000	2 ##	-0.3243	-0.8505	1.1257	0.0000	0.0000	0.0000
40.000	2 ##	-2.4172	-2.7155	0.1997	0.0000	0.0000	0.0000
40.000	3 ##	-0.1828	-0.2077	0.1997	0.0000	0.0000	0.0000
70.000	3 ##	-0.4084	-0.4166	0.0554	0.0000	0.0000	0.0000
70.000	4 ##	0.0090	0.0082	0.0554	0.0000	0.0000	0.0000
9070.000	4 ##	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

depth	layer	s1	s2	s3	eps1	eps2	eps3
0.000	1 ##	8.8259	3.0499	0.0000	0.2084E-04	0.4094E-05	-.4750E-05
20.000	1 ##	1.1255	-2.0893	-4.3575	0.5392E-05	-.3930E-05	-.1051E-04
20.000	2 ##	1.1257	-0.3243	-0.8505	0.1419E-04	-.3931E-05	-.1051E-04
40.000	2 ##	0.1997	-2.4172	-2.7155	0.1483E-04	-.1788E-04	-.2161E-04
40.000	3 ##	0.1997	-0.1828	-0.2077	0.3949E-04	-.1788E-04	-.2161E-04
70.000	3 ##	0.0554	-0.4084	-0.4166	0.4679E-04	-.2278E-04	-.2401E-04
70.000	4 ##	0.0554	0.0090	0.0082	0.4679E-04	-.2278E-04	-.2401E-04
9070.000	4 ##	0.0000	0.0000	0.0000	0.0000E+00	0.0000E+00	0.0000E+00

depth	layer	u(x)	v(y)	w(z)	ex	ey	ez
0.000	1 ##	0.2272E-18	-.1565E-10	-.8979E-02	0.4094E-05	0.2084E-04	-.4750E-05
20.000	1 ##	-.4354E-19	0.7889E-11	-.8961E-02	-.3930E-05	-.1051E-04	0.5392E-05
20.000	2 ##	-.3448E-18	0.7889E-11	-.8961E-02	-.3931E-05	-.1051E-04	0.1419E-04
40.000	2 ##	-.7092E-18	0.1622E-10	-.8698E-02	-.1788E-04	-.2161E-04	0.1483E-04
40.000	3 ##	-.7092E-18	0.1622E-10	-.8698E-02	-.1788E-04	-.2161E-04	0.3949E-04
70.000	3 ##	-.7879E-18	0.1803E-10	-.7520E-02	-.2278E-04	-.2401E-04	0.4679E-04
70.000	4 ##	-.7879E-18	0.1803E-10	-.7520E-02	-.2278E-04	-.2401E-04	0.4679E-04
9070.000	4 ##	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00

Utilized symbols:

sx	: normal stress in the x-direction
sy	: normal stress in the y-direction
sz	: normal stress in the z-direction
tyz	: shear stress in the yz plane, parallel to y or z
txz	: shear stress in the xz plane, parallel to x or z
txy	: shear stress in the xy plane, parallel to x or y
s1	: maximum principal stress
s2	: medium principal stress
s3	: minimum principal stress
eps1	: principal strain
eps2	: principal strain
eps3	: principal strain
u(x)	: displacement in the x-direction
v(y)	: displacement in the y-direction
w(z)	: displacement in the z-direction
ex	: strain in the x-direction
ey	: strain in the y-direction
ez	: strain in the z-direction

The normal stresses are taken positive when they produce compression and negative when they produce tension.